Ατομική Διπλωματική Εργασία

ΕΠΕΚΤΑΣΕΙΣ ΣΤΗΝ ΕΙΚΑΣΙΑ ΤΗΣ ΠΛΗΡΩΣ ΜΙΚΤΗΣ ΙΣΟΡΡΟΠΙΑΣ NASH ΩΣ ΣΕΝΑΡΙΟ ΧΕΙΡΙΣΤΗΣ ΠΕΡΙΠΤΩΣΗΣ : ΤΡΕΙΣ ΠΑΙΚΤΕΣ ΚΑΙ ΑΥΘΑΙΡΕΤΟΣ ΑΡΙΘΜΟΣ ΣΥΝΔΕΣΕΩΝ

Μάριος Μίτας

ΠΑΝΕΠΙΣΤΗΜΙΟ ΚΥΠΡΟΥ

ΤΜΗΜΑ ΠΛΗΡΟΦΟΡΙΚΗΣ

Μάιος 2010

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Επιβλέπων Καθηγητής

Μάριος Μαυρονικόλας

Η Ατομική Διπλωματική Εργασία υποβλήθηκε προς μερική εκπλήρωση των απαιτήσεων απόκτησης του πτυχίου Πληροφορικής του Τμήματος Πληροφορικής του Πανεπιστημίου Κύπρου

Μάιος 2010

Ευχαριστίες

Θα ήθελα να ευχαριστήσω θερμά τον επιβλέποντα καθηγητή μου κύριο Μάριο Μαυρονικόλα για την υποστήριξη , την καθοδήγηση και την μεγάλη βοήθεια που μου πρόσφερε καθ’ όλη την διάρκεια της εκπόνησης της διπλωματικής μου εργασίας, δίνοντας μου σημαντικές συμβουλές έτσι ώστε να καταστεί δυνατή η ολοκλήρωση της εργασίας αυτής.

Ακόμα θα ήθελα να ευχαριστήσω την οικογένεια μου για την στήριξη που μου παρείχε όλο αυτό τον καιρό για την εκπόνηση της διπλωματικής μου εργασίας.

Περίληψη

Στην εργασία αυτή μελετήθηκε η Ισορροπία Nash σε ένα παιχνίδι δρομολόγησης όπου έχουμε τρεις παίκτες και ένα αυθαίρετο αριθμό από συνδέσεις. Οι παίκτες επιλέγουν έναν αριθμό από συνδέσεις με κάποια πιθανότητα για την κάθε σύνδεση. Το Κοινωνικό Κόστος μιας Ισορροπίας Nash ορίζεται ως η αναμενόμενη μέγιστη καθυστέρηση. Σ' αυτή την εργασία μελετούμε την χείριστη περίπτωση Ισορροπίας Nash όπου η τιμή του Κοινωνικού Κόστους μεγιστοποιείται. Συγκεκριμένα, μελετούμε την Εικασία της Πλήρως Μικτής Ισορροπίας Nash η οποία δηλώνει ότι η Χείριστη Περίπτωση Ισορροπίας Nash είναι η Πλήρως Μικτή Ισορροπία Nash, στην οποία κάθε παίκτης επιλέγει κάθε σύνδεση με αυστηρά θετική πιθανότητα.

Περιεχόμενα

Κεφάλαιο 1

Εισαγωγή

1.1 Γενικά

Η εργασία αυτή στηρίζεται στην δημοσίευση **[1]** όπου αποδείκτηκε η ορθότητα της εικασίας της Πλήρως Μικτής Ισορροπίας Nash στην περίπτωση των δυο ταυτόσημων παικτών και αυθαίρετο αριθμό συσχετιζόμενων συνδέσεων στο παιχνίδι δρομολόγησης. Αυτή η εργασία αποτελεί επέκταση της **[1]**, όπου θα μελετήσουμε την περίπτωση με τρείς ταυτόσημους παίκτες και αυθαίρετο αριθμό συσχετιζόμενων συνδέσεων στο εν λόγω παιχνίδι δρομολόγησης και θα προσπαθήσουμε να αποδείξουμε την ισχύ του θεωρήματος. Πιο κάτω θα δούμε τους όρους και τις διάφορες έννοιες που έχουν χρησιμοποιηθεί σ’ αυτή την μελέτη.

Αρχίζουμε με τον ορισμό της Ισορροπίας Nash η οποία αποτελεί και το κύριο σημείο της εργασίας αυτής, ενώ παράλληλα είναι και μια από τις κεντρικές έννοιες της Θεωρίας Παιγνίων. Η Ισορροπία Nash είναι μια σταθερή κατάσταση κατά την διαικπεραίωση ενός στρατηγικού παιχνιδιού, στην οποία οποιοσδήποτε παίκτης προσπαθήσει να αλλάξει στρατηγική, αυτή η αλλαγή θα του επιφέρει κέρδος μικρότερο ή ίσο από αυτό που ήδη έχει.

Συνεχίζουμε με τον ορισμό του παιχνιδιού δρομολόγησης στο οποίο μοντελοποιείται η «εγωιστική» δρομολόγηση. Ο επιθετικός αυτός προσδιορισμός οφείλεται στην εγωιστική συμπεριφορά που παρουσιάζουν οι παίκτες σε ένα μη συνεργάσιμο παιχνίδι όπως το Διαδίκτυο, στην προσπάθεια τους να μειώσουν το αναμενόμενο τους κόστος. Συγκεκριμένα, ο κάθε ένας από τους n παίκτες εφαρμόζει μια *μικτή στρατηγική*, η οποία είναι μια κατανομή πιθανοτήτων σε m συνδέσεις. Για κάθε σύνδεση, η *χωρητικότητα* προσδιορίζει το ρυθμό στον οποίο η σύνδεση επεξεργάζεται την συμφόρηση.

Το Κοινωνικό Κόστος μιας Ισορροπίας Nash είναι η αναμενόμενη τιμή από όλες τις τυχαίες επιλογές των χρηστών της μέγιστης καθυστέρησης για όλες τις συνδέσεις. Η *Χείριστη Περίπτωση Ισορροπίας Nash* είναι αυτή που μεγιστοποιεί την τιμή του Κοινωνικού Κόστους. Στην *Πλήρως Μικτή Ισορροπία Nash κάθε* παίκτης επιλέγει *κάθε* σύνδεση με θετική πιθανότητα**[2]**. Θα δηλώσουμε με F την *Πλήρως Μικτή Ισορροπία Nash* και θα αναφερόμαστε σ' αυτήν ως ΠΜΙΝ.

1.2 Η εικασία της Πλήρως Μικτής Ισορροπίας Nash

Θεωρούμε το μοντέλο των αυθέρετων συμφορίσεων και συνδέσεων. Τότε, για κάθε διάνυσμα συμφόρησης **w** τέτοιο ώστε η Πλήρως Μικτή Ισορροπία Nash **F** υπάρχει, τότε για κάθε Ισορροπία Nash **P** θα έχουμε $\mathrm{SC}(\mathrm{w}, \mathrm{P}) \leq \mathrm{SC}(\mathrm{w}, \mathrm{F})$**[1].**

Η διαισθητικότητα και η φυσικότητα της εικασίας είναι προφανής. Για να υποστηρίξουμε την διαίσθηση, παρατηρούμε ότι η Πλήρως Μικτή Ισορροπία Nash ευνοεί τις συγκρούσεις μεταξύ διαφορετικών παικτών(αφού κάθε παίκτης προσθέτει συμφόρηση στην τρέχων κυκλοφορία αφού επιλέγει με θετική πιθανότητα κάθε σύνδεση). Κατά συνέπεια, αυτή η ενισχυμένη πιθανότητα συγκρούσεων ευνοεί την αντίστοιχη αύξηση στην αναμενόμενη μέγιστη τιμή καθυστέρησης από όλες τις συνδέσεις που είναι και ο ορισμός του Κοινωνικού Κόστους.

Κεφάλαιο 2

Πλαίσιο Εργασίας

2.1 Ορισμοί και Έννοιες

Οι ορισμοί και υποθέσεις που ακολουθούν είναι βασισμένοι στην εργασία **[1]**.

Θα ορίσουμε για κάθε ακέραιο $m \geq 1$, $[m] = \{1, \ldots, m\}$. Θεωρούμε ένα απλό *δίκτυο* αποτελούμενο από ένα σύνολο από m παράλληλες συνδέσεις $1,2, \ldots, m$ από ένα πηγαίο κόμβο σε ένα κόμβο προορισμού. Κάθε ένας από τους n χρήστες $1,2 \ldots, n$ επιθυμεί να δρομολογήσει ένα συγκεκριμένο ποσό συμφόρησης σε μια σύνδεση από την πηγή στον προορισμό. Υποθέτουμε πως $m \geq 2$ και $n \geq 2$.

Δηλώνουμε ως w_i την συμφόρηση του χρήστη $i \in [n]$. Στην περίπτωση μας σημειώνουμε πως όλες οι συμφορήσεις ισούνται με 1. Θα δηλώσουμε με c^{ℓ} την χωρητικότητα της σύνδεσης $\ell \in [m]$, η οποία αναπαριστά το ρυθμό με τον οποίο η σύνδεση επεξεργάζεται την συμφόρηση. Υποθέτουμε χωρίς βλάβη της γενικότητας ότι $c^1 \geq c^2 \geq \cdots \geq c^m$. Επίσης δηλώνουμε την *ολική χωρητικότητα* με $C = \sum_{j \in [m]} c^j$.

2.2 Αναθέσεις και στρατηγικές

Μια μικτή στρατηγική για τον χρήστη $i \in [n]$ είναι μια κατανομή πιθανοτήτων στις συνδέσεις. Μια μικτή ανάθεση είναι ένας πιθανοτικός πίνακας $n \times m$ **P** από nm πιθανότητες p_i^j, $i \in [n]$, $j \in [m]$, όπου p_i^j είναι η πιθανότητα ο χρήστης i να επιλέξει τη σύνδεση j. Αυτή η μικτή ανάθεση **P** είναι πλήρως μικτή αν για όλους τους χρήστες $i \in [n]$ και συνδέσεις $j \in [m]$, $p_i^j > 0$.

Η υποστήριξη του χρήστη $i \in [n]$ στην μικτή ανάθεση **P**, δηλωμένη ως $support_p(i)$, είναι το σύνολο των συνδέσεων $j \in [m]$, για το οποίο ισχύει ότι $p_i^j > 0$.

2.3 Κόστη

Ορίζουμε το *Αναμενόμενο Ατομικό Κόστος* για το χρήστη $i \in [n]$, ως

$$IC_i(P) = \sum_{j \in [m]} p_i^j IC_i^j(P)$$

όπου $IC_i^j(P)$ η αναμενόμενη υπό όρους τιμή του ατομικού κόστους του χρήστη $i \in [n]$ στην μικτή ανάθεση **P** όπου ανατίθεται στην σύνδεση ℓ και ορίζεται ως

$$IC_i^\ell(P) = \frac{1 + \sum_{k \in [n]: k \neq i} p_k^\ell}{c^\ell}$$

2.4 Ισορροπία Nash

Σ' αυτό το σημείο δίνουμε ένα πιο ακριβή ορισμό της Ισορροπίας Nash. Η μικτή ανάθεση **P** είναι μια *Ισορροπία Nash*, αν για κάθε χρήστη $i \in [n]$, ελαχιστοποιεί το *Αναμενόμενο Ατομικό Κόστος* $IC_i(P)$, από όλες τις μικτές αναθέσεις Q που διαφέρουν από την P, αντίστοιχα με τη μικτή στρατηγική του χρήστη i, δηλαδή για όλες αυτές τις μικτές αναθέσεις έχουμε $IC_i(P) \leq IC_i(Q)$. Επομένως, σε μια Ισορροπία Nash, δεν υπάρχει κίνητρο για οποιοδήποτε παίκτη να αποκλείνει από την μικτή στρατηγική του**[1]**.

Εφόσον $\mathrm{IC_i(P)} = \sum_{\mathrm{j}\in[\mathrm{m}]} \mathrm{p_i^j IC_i^j(P)}$, ο πιο πάνω ορισμός της Ισορροπίας Nash μας δηλώνει ότι για κάθε σύνδεση $\mathrm{j} \in [\mathrm{m}]$ τέτοια ώστε $\mathrm{p_i^j} > 0$ και για κάθε σύνδεση $\mathrm{j'} \in [\mathrm{m}]$, τότε είτε θα ισχύει $\mathrm{IC_i^j}$=$\mathrm{IC_i^{j'}}$ αν $\mathrm{p_i^{j'}} > 0$ ή $\mathrm{IC_i^j} \leq \mathrm{IC_i^{j'}}$ αν $\mathrm{p_i^{j'}} = 0$[1].

Η Πλήρως Μικτή Ισορροπία Nash όποτε υπάρχει είναι μοναδική αλλά οι συνδέσεις δεν είναι ισοπίθανες. Για κάθε χρήστη $i \in [n]$, κάθε σύνδεση $\ell \in [\mathrm{m}]$ επιλέγεται με πιθανότητα $f_i^{\ell} = \frac{m+n-1}{(n-1)C} c^l - \frac{1}{n-1}$ ανεξάρτητα του i. Άρα, δηλώνουμε αυτή την κοινή πιθανότητα f_i^{ℓ} ως f^{ℓ}, όπου $\ell \in [m]$.

Κεφάλαιο 3

Τρεις Ταυτόσημοι Παίκτες και Συσχετιζόμενες Συνδέσεις

3.1 Παρουσίαση Θεωρήματος

Θεώρημα 3.1 *Θεωρούμε την περίπτωση των ταυτόσημων χρηστών και των συσχετιζόμενων συνδέσεων, με* $n = 3$. *Τότε η εικασία ΠΜΙΝ ισχύει.*

Υποθέτουμε ότι η Πλήρως Μικτή Ισορροπία Nash υπάρχει και θέτουμε με $n = 3$ στην έκφραση για την f^ℓ, όπου $\ell \in [m]$ και έχουμε

$$f^\ell = \frac{m+2}{2C}c^\ell - \frac{1}{2} = \frac{(m+2)c^\ell - C}{2C}.$$

Εφόσον $f^\ell > 0$ ισχύει ότι $C < (m+2)c^\ell$. Θεωρούμε επίσης οποιαδήποτε αυθαίρετη Ισορροπία Nash $\mathbf{P} = \left(p_i^\ell\right)_{i\in[n],\ell\in[m]}$. Δηλώνουμε $\Delta = SC(F) - SC(P)$. Θα προσπαθήσουμε να αποδείξουμε ότι $\Delta \geq 0$.

3.2 Η αυθαίρετη Ισορροπία Nash και η Δομή

Πρόταση 3.2 (Δομή αυθαίρετης Ισορροπίας Nash): Για την περίπτωση των ταυτόσημων χρηστών και συσχετιζόμενων συνδέσεων θεωρούμε οποιαδήποτε αυθαίρετη Ισορροπία Nash $\boldsymbol{P} \neq \boldsymbol{F}$. Τότε

1. Τα supports των χρηστών είναι $support(1) = [r] \cup I_{12} \cup I_{13} \cup I_1$, $support(2) = [r] \cup I_{12} \cup I_{23} \cup I_2$ και $support(3) = [r] \cup I_{13} \cup I_{23} \cup I_3$, όπου τα $I_1, I_2, I_3, I_{12}, I_{13}$ και I_{23} είναι ξένα μεταξύ τους έτσι ώστε να ισχύει $[r] \cup I_{12} \cup I_{13} \cup I_{23} \cup I_1 \cup I_2 \cup I_3 = [r + |I_{12}| + |I_{13}| + |I_{23}| + |I_1| + |I_2| + |I_3|]$

2. Όλες οι συνδέσεις στο I_1(αντίστοιχα στα I_2 και I_3) έχουν την ίδια χωρητικότητα.

3. Όλες οι συνδέσεις στα I_{12}(αντίστοιχα στα I_{13} και I_{23}) έχουν διαφορετική χωρητικότητα και μεγαλύτερη από αυτές στα I_1, I_2 και I_3

4. Όλες οι συνδέσεις στο $[r]$ έχουν διαφορετική χωρητικότητα και μεγαλύτερη από αυτές στα I_{12}, I_{13} και I_{23}.

Απόδειξη: Αποκλείουμε τις ακόλουθες περιπτώσεις :

1. <u>$support(1) = support(2) = support(3) = [m]$</u>

Εφόσον $\boldsymbol{P} \neq \boldsymbol{F}$ αυτή η περίπτωση εξαιρείται.

2. <u>$support(1) = support(2) = support(3) \subset [m]$</u>

Παίρνουμε οποιοδήποτε ζευγάρι συνδέσεις $\ell \in support(1)$ και $\ell' \in [m] \setminus support(1)$. Από τον ορισμό του Αναμενόμενου Ατομικού Κόστους, ισχύει ότι $IC_1^{\ell} \geq \frac{1}{c^{\ell}}$ και $IC_1^{\ell'} = \frac{1}{c^{\ell'}}$. Αφού P είναι Ισορροπία Nash έχουμε ότι $IC_1^{\ell} \leq IC_1^{\ell'}$. Επομένως $c^{\ell} \geq c^{\ell'}$. Άρα έχουμε $support(1) = support(2) = support(3) = [m - s]$ όπου $s \geq 1$.

Ισχύει

$$\begin{aligned}
& IC_1(P) \\
\leq\ & IC_1^{\ell'}(P) \text{ (εφόσον P είναι Ισορροπία Nash)} \\
=\ & \frac{1}{c^{\ell'}} \\
<\ & \frac{1 + 2\mathrm{f}^{\ell'}}{\mathrm{c}^{\ell'}} \text{ (εφόσον } f^{\ell'} > 0) \\
=\ & IC_1^{\ell'}(F) \\
=\ & IC_1(F) \text{(εφόσον F είναι πλήρως μικτή Ισορροπία Nash)}
\end{aligned}$$

Επίσης, P είναι μια πλήρως μικτή Ισορροπία Nash στο $[m-s]$, έτσι ώστε για κάθε χρήστη $i \in [3]$ και σύνδεση $\ell \in [m-s]$, ισχύει ότι $p_i^\ell = \frac{m-s+2}{2\sum_{j\in[m-s]} c^j} c^\ell - \frac{1}{2}$. Ακόμα έχουμε ότι $f^\ell = \frac{m+2}{2C} c^\ell - \frac{1}{2}$. Έτσι

$$p_i^\ell - f^\ell$$

$$= \quad \frac{m-s+2}{2\sum_{j\in[m-s]} c^j} c^\ell - \frac{1}{2} - \frac{m+2}{2C} c^\ell + \frac{1}{2}$$

$$= \quad c^\ell \left(\frac{m-s+2}{2\sum_{j\in[m-s]} c^j} - \frac{m+2}{2C} \right)$$

$$= \quad c^\ell \left(\frac{m+2}{2\sum_{j\in[m-s]} c^j} - \frac{s}{2\sum_{j\in[m-s]} c^j} - \frac{m+2}{2C} \right)$$

$$= \quad \frac{c^\ell}{2C\sum_{j\in[m-s]} c^j} \left((m+2)C - sC - (m+2) \sum_{j\in[m-s]} c^j \right)$$

$$= \quad \frac{c^\ell}{2C\sum_{j\in[m-s]} c^j} \left((m+2)(C - \sum_{j\in[m-s]} c^j) - sC \right)$$

$$= \quad \frac{c^\ell}{2C\sum_{j\in[m-s]} c^j} \left(\sum_{j\in[m]\setminus[m-s]} (m+2)c^j - sC \right)$$

$$> \quad \frac{c^\ell}{2C\sum_{j\in[m-s]} c^j} (sC - sC) \quad \left(\text{εφόσον } C < (m+2)c^\ell \right)$$

$$= 0.$$

Άρα $\boldsymbol{p_i^\ell} > \boldsymbol{f^\ell}$.

Επομένως,

$$
\begin{aligned}
& IC_1(P) \\
= \ & IC_1^{\ell}(P) \text{ (εφόσον } \ell \in support(1) \text{ και P είναι Ισορροπία Nash)} \\
= \ & \frac{1 + p_2^{\ell} + p_3^{\ell}}{c^{l}} \\
> \ & \frac{1 + 2f^{\ell}}{\mathrm{c}^{\ell}} \text{ (εφόσον } p_i^{\ell} > f^{\ell}) \\
= \ & IC_1^{\ell}(F) \\
= \ & IC_1(F) \text{(εφόσον F είναι πλήρως μικτή Ισορροπία Nash)}
\end{aligned}
$$

το οποίο είναι αντίφαση. Επομένως, αυτή η περίπτωση εξαιρείται.

3. $\underline{support(2) = support(1)}$ $\underline{\text{και}}$ $\underline{support(1) \cap support(3) \neq \emptyset}$

Παίρνουμε οποιοδήποτε ζευγάρι συνδέσεις $\mathrm{j} \in \mathrm{support(1)}$ και $\mathrm{k} \in [\mathrm{m}]\backslash\mathrm{support(1)}\backslash \mathrm{support(2)}\backslash\mathrm{support(3)}$. Εφόσον P και F είναι Ισορροπία Nash, τότε $\mathrm{IC_1(P)} = \mathrm{IC_1^j(P)} \leq \mathrm{IC_1^k(P)}$ και $\mathrm{IC_1(F)} = \mathrm{IC_1^j(F)} = \mathrm{IC_1^k(F)}$. Επίσης ισχύει $\mathrm{IC_1^j(P)} = \frac{1+\mathrm{p_2^j}+\mathrm{p_3^j}}{\mathrm{c^j}}$, $\mathrm{IC_1^k(P)} = \frac{1}{\mathrm{c^k}}$ και $\mathrm{IC_3^j(F)} = \frac{1+2\mathrm{f^j}}{\mathrm{c^j}}$, $\mathrm{IC_3^k(F)} = \frac{1+2\mathrm{f^k}}{\mathrm{c^k}}$. Άρα $\frac{1+\mathrm{p_2^j}+\mathrm{p_3^j}}{\mathrm{c^j}} \leq \frac{1}{\mathrm{c^k}}$ και $\frac{1+2\mathrm{f^j}}{\mathrm{c^j}} = \frac{1+2\mathrm{f^k}}{\mathrm{c^k}}$. Επομένως $\frac{1+\mathrm{p_2^j}+\mathrm{p_3^j}}{\mathrm{c^j}} \leq \frac{1+2\mathrm{f^j}}{\mathrm{c^j}} \Rightarrow \mathrm{p_2^j} + \mathrm{p_3^j} \leq 2\mathrm{f^j}$. Αντίφαση.

Οι συμμετρικές περιπτώσεις όπου $\mathrm{support(3)} = \mathrm{support(1)}$ και $\mathrm{support(1)} \cap \mathrm{support(2)} \neq \emptyset$ και, $\mathrm{support(3)} = \mathrm{support(2)}$ και $\mathrm{support(3)} \cap \mathrm{support(1)} \neq \emptyset$ εξαιρούνται με τον ίδιο τρόπο.

4. $\underline{support(2) \subset support(1) = support(3)}$

Παίρνουμε οποιοδήποτε ζευγάρι συνδέσεις $\mathrm{j} \in \mathrm{support(2)}$ και $\mathrm{k} \in \mathrm{support(1)}\backslash \mathrm{support(2)}$. Εφόσον P και F είναι Ισορροπία Nash, τότε $\mathrm{IC_1(P)} = \mathrm{IC_1^j(P)} = \mathrm{IC_1^k(P)}$ και $\mathrm{IC_1(F)} = \mathrm{IC_1^j(F)} = \mathrm{IC_1^k(F)}$. Επίσης ισχύει $\mathrm{IC_1^j(P)} = \frac{1+\mathrm{p_2^j}+\mathrm{p_3^j}}{\mathrm{c^j}}$, $\mathrm{IC_1^k(P)} = \frac{1}{\mathrm{c^k}}$ και $\mathrm{IC_1^j(F)} = \frac{1+2\mathrm{f^j}}{\mathrm{c^j}}$, $\mathrm{IC_1^k(F)} = \frac{1+2\mathrm{f^k}}{\mathrm{c^k}}$. Άρα $\frac{1+\mathrm{p_2^j}+\mathrm{p_3^j}}{\mathrm{c^j}} = \frac{1}{\mathrm{c^k}}$ και $\frac{1+2\mathrm{f^j}}{\mathrm{c^j}} = \frac{1+2\mathrm{f^k}}{\mathrm{c^k}}$. Επομένως $\frac{1+\mathrm{p_2^j}+\mathrm{p_3^j}}{\mathrm{c^j}} \leq \frac{1+2\mathrm{f^j}}{\mathrm{c^j}} \Rightarrow \mathrm{p_2^j} + \mathrm{p_3^j} \leq 2\mathrm{f^j}$. Αντίφαση.

Οι συμμετρικές περιπτώσεις όπου $\text{support}(1) \subset \text{support}(2) = \text{support}(3)$, $\text{support}(3) \subset \text{support}(1) = \text{support}(2)$, $\text{support}(1) \subset \text{support}(3) = \text{support}(2)$, $\text{support}(2) \subset \text{support}(3) = \text{support}(1)$ και $\text{support}(3) \subset \text{support}(2) = \text{support}(1)$ εξαιρούνται με τον ίδιο τρόπο.

5. $\underline{support(2) = support(3) \subset support(1)}$

Παίρνουμε οποιοδήποτε ζευγάρι συνδέσεις $j \in \text{support}(2)$ και $k \in \text{support}(1) \setminus \text{support}(2)$. Εφόσον P και F είναι Ισορροπία Nash, τότε $\text{IC}_1(\text{P}) = \text{IC}_1^j(\text{P}) = \text{IC}_1^k(\text{P})$ και $\text{IC}_1(\text{F}) = \text{IC}_1^j(\text{F}) = \text{IC}_1^k(\text{F})$. Επίσης ισχύει $\text{IC}_1^j(\text{P}) = \frac{1+p_2^j+p_3^j}{c^j}$, $\text{IC}_1^k(\text{P}) = \frac{1}{c^k}$ και $\text{IC}_1^j(\text{F}) = \frac{1+2f^j}{c^j}$, $\text{IC}_1^k(\text{F}) = \frac{1+2f^k}{c^k}$. Άρα $\frac{1+p_2^j+p_3^j}{c^j} = \frac{1}{c^k}$ και $\frac{1+2f^j}{c^j} = \frac{1+2f^k}{c^k}$. Επομένως $\frac{1+p_2^j+p_3^j}{c^j} \leq \frac{1+2f^j}{c^j} \Rightarrow p_2^j + p_3^j \leq 2f^j$. Αντίφαση.

Οι συμμετρικές περιπτώσεις όπου $\text{support}(1) = \text{support}(2) \subset \text{support}(3)$ και $\text{support}(1) = \text{support}(3) \subset \text{support}(2)$ εξαιρούνται με τον ίδιο τρόπο.

6. $\underline{support(2) \subset support(1)}$ <u>και</u> $\underline{support(1) \cap support(3) \neq \emptyset}$

Παίρνουμε οποιοδήποτε ζευγάρι συνδέσεις $j \in \text{support}(2)$ και $k \in \text{support}(3) \setminus \text{support}(2)$. Εφόσον P και F είναι Ισορροπία Nash, τότε $\text{IC}_1(\text{P}) = \text{IC}_1^j(\text{P}) = \text{IC}_1^k(\text{P})$ και $\text{IC}_1(\text{F}) = \text{IC}_1^j(\text{F}) = \text{IC}_1^k(\text{F})$. Επίσης ισχύει $\text{IC}_1^j(\text{P}) = \frac{1+p_2^j+p_3^j}{c^j}$, $\text{IC}_1^k(\text{P}) = \frac{1}{c^k}$ και $\text{IC}_1^j(\text{F}) = \frac{1+2f^j}{c^j}$, $\text{IC}_1^k(\text{F}) = \frac{1+2f^k}{c^k}$. Άρα $\frac{1+p_2^j+p_3^j}{c^j} = \frac{1}{c^k}$ και $\frac{1+2f^j}{c^j} = \frac{1+2f^k}{c^k}$. Επομένως $\frac{1+p_2^j+p_3^j}{c^j} \leq \frac{1+2f^j}{c^j} \Rightarrow p_2^j + p_2^j \leq 2f^j$. Αντίφαση.

Οι συμμετρικές περιπτώσεις όπου $\text{support}(1) \subset \text{support}(2)$ και $\text{support}(2) \cap \text{support}(3) \neq \emptyset$, $\text{support}(2) \subset \text{support}(3)$ και $\text{support}(3) \cap \text{support}(1) \neq \emptyset$, $\text{support}(3) \subset \text{support}(2)$ και $\text{support}(2) \cap \text{support}(1) \neq \emptyset$, $\text{support}(1) \subset \text{support}(3)$ και $\text{support}(3) \cap \text{support}(2) \neq \emptyset$ και $\text{support}(3) \subset \text{support}(1)$ και $\text{support}(1) \cap \text{support}(2) \neq \emptyset$, εξαιρούνται με τον ίδιο τρόπο.

7. $\underline{support(1) \cap support(2) \neq \emptyset}$ <u>και</u> $\underline{support(1) \cap support(3) = \emptyset}$ <u>και</u> $\underline{support(2) \cap support(3) = \emptyset}$

Παίρνουμε οποιοδήποτε ζευγάρι συνδέσεις $\mathrm{j} \in \mathrm{support}(1)$ και $\mathrm{k} \in \mathrm{support}(3) \setminus \mathrm{support}(2)$. Εφόσον P και F είναι Ισορροπία Nash, τότε $\mathrm{IC}_3(\mathrm{P}) = \mathrm{IC}_3^{\mathrm{j}}(\mathrm{P}) = \mathrm{IC}_3^{\mathrm{k}}(\mathrm{P})$ και $\mathrm{IC}_3(\mathrm{F}) = \mathrm{IC}_3^{\mathrm{j}}(\mathrm{F}) = \mathrm{IC}_3^{\mathrm{k}}(\mathrm{F})$. Επίσης ισχύει $\mathrm{IC}_3^{\mathrm{j}}(\mathrm{P}) = \frac{1+\mathrm{p}_1^{\mathrm{j}}+\mathrm{p}_2^{\mathrm{j}}}{\mathrm{c}^{\mathrm{j}}}$, $\mathrm{IC}_3^{\mathrm{k}}(\mathrm{P}) = \frac{1}{\mathrm{c}^{\mathrm{k}}}$ και $\mathrm{IC}_3^{\mathrm{j}}(\mathrm{F}) = \frac{1+2\mathrm{f}^{\mathrm{j}}}{\mathrm{c}^{\mathrm{j}}}$, $\mathrm{IC}_3^{\mathrm{k}}(\mathrm{F}) = \frac{1+2\mathrm{f}^{\mathrm{k}}}{\mathrm{c}^{\mathrm{k}}}$. Άρα $\frac{1+\mathrm{p}_1^{\mathrm{j}}+\mathrm{p}_2^{\mathrm{j}}}{\mathrm{c}^{\mathrm{j}}} = \frac{1}{\mathrm{c}^{\mathrm{k}}}$ και $\frac{1+2\mathrm{f}^{\mathrm{j}}}{\mathrm{c}^{\mathrm{j}}} = \frac{1+2\mathrm{f}^{\mathrm{k}}}{\mathrm{c}^{\mathrm{k}}}$. Επομένως $\frac{1+\mathrm{p}_1^{\mathrm{j}}+\mathrm{p}_2^{\mathrm{j}}}{\mathrm{c}^{\mathrm{j}}} \leq \frac{1+2\mathrm{f}^{\mathrm{j}}}{\mathrm{c}^{\mathrm{j}}} \Rightarrow \mathrm{p}_1^{\mathrm{j}} + \mathrm{p}_2^{\mathrm{j}} \leq 2\mathrm{f}^{\mathrm{j}}$. Αντίφαση.

Οι συμμετρικές περιπτώσεις όπου $\mathrm{support}(2) \cap \mathrm{support}(3) \neq \emptyset$ και $\mathrm{support}(3) \cap \mathrm{support}(1) = \emptyset$ και $\mathrm{support}(2) \cap \mathrm{support}(1) = \emptyset$, και $\mathrm{support}(3) \cap \mathrm{support}(1) \neq \emptyset$ και $\mathrm{support}(1) \cap \mathrm{support}(2) = \emptyset$ και $\mathrm{support}(3) \cap \mathrm{support}(2) = \emptyset$, εξαιρούνται με τον ίδιο τρόπο.

8. <u>$support(1) \cap support(2) \neq \emptyset$ και $support(1) \cap support(3) \neq \emptyset$ και $support(2) \cap support(3) = \emptyset$</u>

Παίρνουμε οποιοδήποτε ζευγάρι συνδέσεις $\mathrm{j} \in \mathrm{support}(1)$ και $\mathrm{k} \in \mathrm{support}(3) \setminus \mathrm{support}(2)$. Εφόσον P και F είναι Ισορροπία Nash, τότε $\mathrm{IC}_3(\mathrm{P}) = \mathrm{IC}_3^{\mathrm{j}}(\mathrm{P}) = \mathrm{IC}_3^{\mathrm{k}}(\mathrm{P})$ και $\mathrm{IC}_3(\mathrm{F}) = \mathrm{IC}_3^{\mathrm{j}}(\mathrm{F}) = \mathrm{IC}_3^{\mathrm{k}}(\mathrm{F})$. Επίσης ισχύει $\mathrm{IC}_3^{\mathrm{j}}(\mathrm{P}) = \frac{1+\mathrm{p}_1^{\mathrm{j}}+\mathrm{p}_2^{\mathrm{j}}}{\mathrm{c}^{\mathrm{j}}}$, $\mathrm{IC}_3^{\mathrm{k}}(\mathrm{P}) = \frac{1}{\mathrm{c}^{\mathrm{k}}}$ και $\mathrm{IC}_3^{\mathrm{j}}(\mathrm{F}) = \frac{1+2\mathrm{f}^{\mathrm{j}}}{\mathrm{c}^{\mathrm{j}}}$, $\mathrm{IC}_3^{\mathrm{k}}(\mathrm{F}) = \frac{1+2\mathrm{f}^{\mathrm{k}}}{\mathrm{c}^{\mathrm{k}}}$. Άρα $\frac{1+\mathrm{p}_1^{\mathrm{j}}+\mathrm{p}_2^{\mathrm{j}}}{\mathrm{c}^{\mathrm{j}}} = \frac{1}{\mathrm{c}^{\mathrm{k}}}$ και $\frac{1+2\mathrm{f}^{\mathrm{j}}}{\mathrm{c}^{\mathrm{j}}} = \frac{1+2\mathrm{f}^{\mathrm{k}}}{\mathrm{c}^{\mathrm{k}}}$. Επομένως $\frac{1+\mathrm{p}_1^{\mathrm{j}}+\mathrm{p}_2^{\mathrm{j}}}{\mathrm{c}^{\mathrm{j}}} \leq \frac{1+2\mathrm{f}^{\mathrm{j}}}{\mathrm{c}^{\mathrm{j}}} \Rightarrow \mathrm{p}_1^{\mathrm{j}} + \mathrm{p}_2^{\mathrm{j}} \leq 2\mathrm{f}^{\mathrm{j}}$. Αντίφαση.

Οι συμμετρικές περιπτώσεις όπου $\mathrm{support}(2) \cap \mathrm{support}(3) \neq \emptyset$ και $\mathrm{support}(3) \cap \mathrm{support}(1) \neq \emptyset$ και $\mathrm{support}(2) \cap \mathrm{support}(1) = \emptyset$, και $\mathrm{support}(3) \cap \mathrm{support}(1) \neq \emptyset$ και $\mathrm{support}(1) \cap \mathrm{support}(2) \neq \emptyset$ και $\mathrm{support}(3) \cap \mathrm{support}(2) = \emptyset$, εξαιρούνται με τον ίδιο τρόπο.

9. <u>Υπάρχουν συνδέσεις $j, k \in support(1) \setminus support(2) \setminus support(3)$ έτσι ώστε $c^j \neq c^k$</u>

Εφόσον P είναι Ισορροπία Nash, τότε $IC_1(P) = IC_1^j(P) = IC_1^k(P)$. Επομένως $\frac{1}{c^j} = \frac{1}{c^k}$. Επομένως $c^j = c^k$, αντίφαση. Επομένως αυτή η περίπτωση εξαιρείται.

Οι συμμετρικές περιπτώσεις όπου υπάρχουν συνδέσεις $\mathrm{j}, \mathrm{k} \in \mathrm{support}(2) \setminus \mathrm{support}(1) \setminus \mathrm{support}(3)$ έτσι ώστε $\mathrm{c}^{\mathrm{j}} \neq \mathrm{c}^{\mathrm{k}}$ και $\mathrm{j}, \mathrm{k} \in \mathrm{support}(3) \setminus \mathrm{support}(2) \setminus \mathrm{support}(1)$ έτσι ώστε $\mathrm{c}^{\mathrm{j}} \neq \mathrm{c}^{\mathrm{k}}$ εξαιρούνται με τον ίδιο τρόπο.

10. α) Υπάρχουν συνδέσεις $j \in support(1) \backslash support(2) \backslash support(3)$ και $k \in support(1) \cap support(2) \cap support(3)$ έτσι ώστε $c^j \geq c^k$

Εφόσον P είναι Ισορροπία Nash, τότε $IC_1(P) = IC_1^j(P) = IC_1^k(P)$. Επομένως $\frac{1}{c^j} = \frac{1+p_2^k+p_3^k}{c^k}$. Αφού $p_2^k + p_3^k > 0$ ισχύει ότι $\frac{1}{c^j} > \frac{1}{c^k}$ άρα $c^j < c^k$, αντίφαση. Επομένως αυτή η περίπτωση εξαιρείται.

Οι συμμετρικές περιπτώσεις όπου υπάρχουν συνδέσεις $\mathrm{j} \in \mathrm{support}(2) \backslash \mathrm{support}(1) \backslash \mathrm{support}(3)$ και $\mathrm{k} \in \mathrm{support}(1) \cap \mathrm{support}(2) \cap \mathrm{support}(3)$ έτσι ώστε $\mathrm{c^j} \geq \mathrm{c^k}$ και $\mathrm{j} \in \mathrm{support}(3) \backslash \mathrm{support}(2) \backslash \mathrm{support}(1)$ και $\mathrm{k} \in \mathrm{support}(1) \cap \mathrm{support}(2) \cap \mathrm{support}(3)$ έτσι ώστε $\mathrm{c^j} \geq \mathrm{c^k}$εξαιρούνται με τον ίδιο τρόπο.

β) Υπάρχουν συνδέσεις $j \in support(1) \backslash support(2) \backslash support(3)$ και $k \in support(1) \cap support(2) \backslash support(3)$ έτσι ώστε $c^j \geq c^k$

Εφόσον P είναι Ισορροπία Nash, τότε $IC_1(P) = IC_1^j(P) = IC_1^k(P)$. Επομένως $\frac{1}{c^j} = \frac{1+p_2^k}{c^k}$. Αφού $p_2^k > 0$ ισχύει ότι $\frac{1}{c^j} > \frac{1}{c^k}$ άρα $c^j < c^k$, αντίφαση. Επομένως αυτή η περίπτωση εξαιρείται.

Οι συμμετρικές περιπτώσεις όπου υπάρχουν συνδέσεις $\mathrm{j} \in \mathrm{support}(2) \backslash \mathrm{support}(1) \backslash \mathrm{support}(3)$ και $\mathrm{k} \in \mathrm{support}(1) \cap \mathrm{support}(2) \backslash \mathrm{support}(3)$ έτσι ώστε $\mathrm{c^j} \geq \mathrm{c^k}$, $\mathrm{j} \in \mathrm{support}(3) \backslash \mathrm{support}(1) \backslash \mathrm{support}(2)$ και $\mathrm{k} \in \mathrm{support}(2) \cap \mathrm{support}(3) \backslash \mathrm{support}(1)$ έτσι ώστε $\mathrm{c^j} \geq \mathrm{c^k}$ και οι υπόλοιπες συμμετρικές περιπτώσεις έτσι ώστε $\mathrm{c^j} \geq \mathrm{c^k}$, εξαιρούνται με τον ίδιο τρόπο.

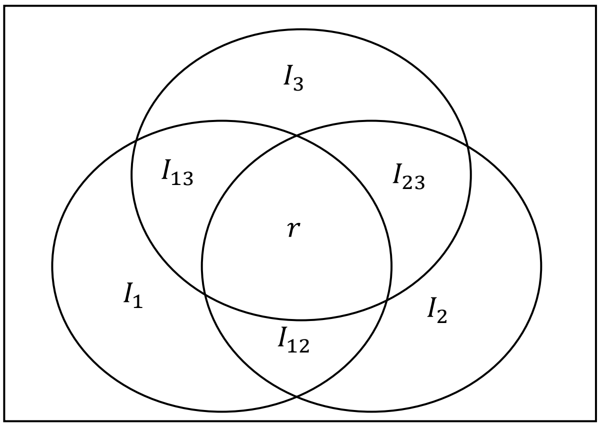
11. Υπάρχουν συνδέσεις $j \in support(1) \backslash support(2) \backslash support(3)$ και $k \in [m] \backslash support(1) \cap support(2) \cap support(3) \backslash support(1) \cap support(2) \backslash support(1) \cap support(3) \backslash support(2) \cap support(3)$ έτσι ώστε $c^j < c^k$

Εφόσον P είναι Ισορροπία Nash, τότε $\mathrm{IC_1(P) = IC_1^j(P)}$ και $\mathrm{IC_1(P) \leq IC_1^k(P)}$. Επομένως $\frac{1}{\mathrm{c^j}} \leq \frac{1}{\mathrm{c^k}}$ ή $\mathrm{c^k \leq c^j}$, αντίφαση. Επομένως αυτή η περίπτωση εξαιρείται.

Οι συμμετρικές περιπτώσεις όπου $\mathrm{j} \in \mathrm{support}(2) \backslash \mathrm{support}(1) \backslash \mathrm{support}(3)$ και $\mathrm{k} \in [\mathrm{m}] \backslash \mathrm{support}(1) \cap \mathrm{support}(2) \cap \mathrm{support}(3) \backslash \mathrm{support}(1) \cap \mathrm{support}(2) \backslash \mathrm{support}(1) \cap \mathrm{support}(3) \backslash \mathrm{support}(2) \cap \mathrm{support}(3)$ έτσι ώστε $\mathrm{c^j} < \mathrm{c^k}$ και $\mathrm{j} \in \mathrm{support}(3) \backslash \mathrm{support}(2) \backslash \mathrm{support}(1)$ και $\mathrm{k} \in [\mathrm{m}] \backslash \mathrm{support}(1) \cap \mathrm{support}(2) \cap$

support(3)\support(1) ∩ support(2)\support(1) ∩ support(3)\support(2) ∩ support(3) έτσι ώστε $c^j < c^k$, εξαιρούνται με τον ίδιο τρόπο.

Οι πιο πάνω εξερομένες περιπτώσεις μας δίνουν τη δομή της αυθαίρετης Ισορροπίας Nash που φαίνεται στην Πρόταση 3.2. Πιο κάτω βλέπουμε πως αναπαριστάται σχηματικά αυτή η δομή, όπου το σύνολο των συνδέσεων χωρίζεται σε τρια σύνολα, το support του χρήστη 1, το support του χρήστη 2 και το support του χρήστη 3 :



Σχήμα 3.1 : Δομή Αυθαίρετης Ισορροπίας Nash

Έστω $m_1, m_2, m_3, m_{12}, m_{13}, m_{23}$ τα μεγέθη των $I_1, I_2, I_3, I_{12}, I_{13}, I_{23}$ αντίστοιχα. Σύμφωνα με την Πρόταση 3.2 προκύπτουν τα ακόλουθα:

$$c^j = \begin{cases} \gamma^j, \alpha\nu\ j \in [r] \\ \zeta^j, \alpha\nu\ j \in I_{12} = [r + m_1 + m_2 + m_3 + m_{12}]\backslash[r + m_1 + m_2 + m_3] \\ \varepsilon^j, \alpha\nu\ j \in I_{13} = [r + m_1 + m_2 + m_3 + m_{12} + m_{13}]\backslash[r + m_1 + m_2 + m_3 + m_{12}] \\ \delta^j, \alpha\nu\ j \in I_{23} = [r + m_1 + m_2 + m_3 + m_{12} + m_{13} + m_{23}]\backslash[r + m_1 + m_2 + m_3 + m_{12} + m_{13}] \\ \gamma, \alpha\nu\ j \in I_1 = [r + m_1]\backslash[r] \\ \beta, \alpha\nu\ j \in I_2 = [r + m_1 + m_2]\backslash[r + m_1] \\ \alpha, \alpha\nu\ j \in I_3 = [r + m_1 + m_2 + m_3]\backslash[r + m_1 + m_2] \end{cases}$$

Χωρίς βλάβη της γενικότητας, υποθέτουμε ότι

$\gamma \geq \beta \geq \alpha$ και $\zeta^{r+m_1+m_2+m_3+1} \geq \cdots \geq \zeta^{r+m_1+m_2+m_3+m_{12}} \geq \varepsilon^{r+m_1+m_2+m_3+m_{12}+1} \geq \cdots \geq \varepsilon^{r+m_1+m_2+m_3+m_{12}+m_{13}} \geq \delta^{r+m_1+m_2+m_3+m_{12}+m_{13}+1} \geq \delta^{r+m_1+m_2+m_3+m_{12}+m_{13}+m_{23}}$.

<u>**Πρόταση 3.3**</u>: Άρα σύμφωνα με την πρόταση 3.2 έχουμε

$\gamma^1 \geq \cdots \geq \gamma^r \geq \zeta^{r+m_1+m_2+m_3+1} \geq \cdots \geq \zeta^{r+m_1+m_2+m_3+m_{12}} \geq \varepsilon^{r+m_1+m_2+m_3+m_{12}+1} \geq \cdots \geq \varepsilon^{r+m_1+m_2+m_3+m_{12}+m_{13}} \geq \delta^{r+m_1+m_2+m_3+m_{12}+m_{13}+1} \geq \delta^{r+m_1+m_2+m_3+m_{12}+m_{13}+m_{23}} \geq \gamma \geq \beta \geq \alpha$

Δηλώνουμε τις ακόλουθες σχέσεις :

$$A = \frac{a(m+2) - C}{aC}$$

$$B = \frac{\beta(m+2) - C}{\beta C}$$

$$G = \frac{\gamma(m+2) - C}{\gamma C}$$

Αφού $a(m+2) > C, \beta(m+2) > C$ και $\gamma(m+2) > C$ ακολουθεί ότι $A > 0, B > 0$ και $G > 0$ και αφού $\gamma \geq \beta \geq \alpha$ ισχύει ότι $G \geq B \geq A$. Επίσης δηλώνουμε τα ακόλουθα :

$$\Gamma = \sum_{j \in [r]} \gamma^j$$

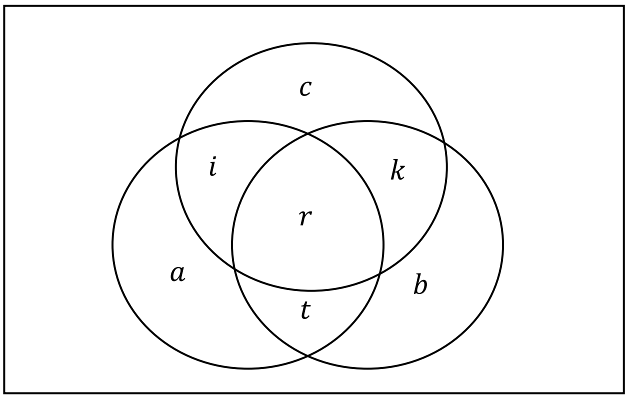
$$Z = \sum_{j \in [r+m_1+m_2+m_3+m_{12}] \setminus [r+m_1+m_2+m_3]} \zeta^j$$

$$E = \sum_{j \in [r+m_1+m_2+m_3+m_{12}+m_{13}] \setminus [r+m_1+m_2+m_3+m_{12}]} \varepsilon^j$$

$$\Delta = \sum_{j \in [r+m_1+m_2+m_3+m_{12}+m_{13}+m_{23}] \setminus [r+m_1+m_2+m_3+m_{12}+m_{13}]} \delta^j$$

3.3 Ανισότητες στην Ισορροπία Nash

Εκμεταλλευόμαστε το γεγονός της *Ισορροπίας Nash* στη μικτή ανάθεση **P** και έχουμε τις πιο κάτω ισότητες και ανισότητες για συνδέσεις $t \in [r + m_1 + m_2 + m_3 + m_{12}] \setminus [r + m_1 + m_2 + m_3]$, $i \in [r + m_1 + m_2 + m_3 + m_{12} + m_{13}] \setminus [r + m_1 + m_2 + m_3 + m_{12}]$, $k \in [r + m_1 + m_2 + m_3 + m_{12} + m_{13} + m_{23}] \setminus [r + m_1 + m_2 + m_3 + m_{12} + m_{13}]$, $j \in [r]$, $a \in [r + m_1] \setminus [r]$, $b \in [r + m_1 + m_2] \setminus [r + m_1]$ και $c \in [r + m_1 + m_2 + m_3] \setminus [r + m_1 + m_2]$, όπως αυτές φαίνονται και στο σχήμα:



Σχήμα A-1.1 : Δομή Αυθαίρετης Ισορροπίας Nash και οι συνδέσεις a,b,c,i,k,t,r σ'αυτήν.

a) Για το χρήστη 1 ισχύει:

$IC_1(P) =$

$IC_1^t(P) = IC_1^i(P) = IC_1^j(P) = IC_1^a(P) \leq IC_1^k(P)$

όπου

$IC_1^t(P) = \frac{1+p_2^t}{\zeta^t}, IC_1^i(P) = \frac{1+p_3^i}{\varepsilon^i}, IC_1^j(P) = \frac{1+p_2^j+p_3^j}{\gamma^j}, IC_1^\alpha(P) = \frac{1}{\gamma}, IC_1^k(P) = \frac{1+p_2^k+p_3^k}{\delta^k}$

Άρα,

$$\boldsymbol{\frac{1+p_2^t}{\zeta^t} = \frac{1+p_3^i}{\varepsilon^i} = \frac{1+p_2^j+p_3^j}{\gamma^j} = \frac{1}{\gamma} \leq \frac{1+p_2^k+p_3^k}{\delta^k}}$$

b) Για το χρήστη 2 ισχύει:

$IC_2(P) =$

$IC_2^k(P) = IC_2^t(P) = IC_2^j(P) = IC_2^b(P) \leq IC_2^i(P)$

όπου

$IC_2^k(P) = \frac{1+p_3^k}{\delta^k}, IC_2^t(P) = \frac{1+p_1^t}{\zeta^t}, IC_2^j(P) = \frac{1+p_1^j+p_3^j}{\gamma^j}, IC_2^b(P) = \frac{1}{\beta}, IC_2^i(P) = \frac{1+p_1^i+p_3^i}{\varepsilon^i}$

Άρα,

$$\boldsymbol{\frac{1+p_3^k}{\delta^k} = \frac{1+p_1^t}{\zeta^t} = \frac{1+p_1^j+p_3^j}{\gamma^j} = \frac{1}{\beta} \leq \frac{1+p_1^i+p_3^i}{\varepsilon^i}}$$

c) Για το χρήστη 3 ισχύει:

$IC_3(P) =$

$IC_3^k(P) = IC_3^i(P) = IC_3^j(P) = IC_3^c(P) \leq IC_3^t(P)$

όπου

$IC_3^k(P) = \frac{1+p_2^k}{\delta^k}, IC_3^i(P) = \frac{1+p_1^i}{\varepsilon^i}, IC_3^j(P) = \frac{1+p_1^j+p_2^j}{\gamma^j}, IC_3^b(P) = \frac{1}{a}, IC_3^t(P) = \frac{1+p_1^t+p_2^t}{\zeta^t}$

Άρα,

$$\boldsymbol{\frac{1+p_2^k}{\delta^k} = \frac{1+p_1^i}{\varepsilon^i} = \frac{1+p_1^j+p_2^j}{\gamma^j} = \frac{1}{a} \leq \frac{1+p_1^t+p_2^t}{\zeta^t}}$$

<u>Πρόταση 3.4</u>: Θα αποδείξουμε ότι

1) i) $max\left\{\frac{2}{c^i},\frac{1}{c^k}\right\}=\frac{2}{c^i}$, όταν $i=j, i<k$

ii)$max\left\{\frac{2}{c^i},\frac{1}{c^k}\right\}=\frac{2}{c^i}$, όταν $i=j, i>k$

2) i) $max\left\{\frac{2}{c^k},\frac{1}{c^i}\right\}=\frac{2}{c^k}$, όταν $j=k, k<i$

ii)$max\left\{\frac{2}{c^k},\frac{1}{c^i}\right\}=\frac{2}{c^k}$, όταν $j=k, k>i$

3) i) $max\left\{\frac{2}{c^k},\frac{1}{c^j}\right\}=\frac{2}{c^k}$, όταν $i=k, i<j$

ii)$max\left\{\frac{2}{c^k},\frac{1}{c^j}\right\}=\frac{2}{c^k}$, όταν $i=k, i>j$

<u>Απόδειξη (1)</u>

Για την απόδειξη του i) θα χρειαστεί να αποδείξουμε τις ακόλουθες περιπτώσεις

$1)\frac{2}{\zeta^t}\geq\frac{1}{a},\quad 2)\frac{2}{\zeta^t}\geq\frac{1}{\varepsilon^i},\quad 3)\frac{2}{\zeta^t}\geq\frac{1}{\delta^k},\quad 4)\frac{2}{\gamma^j}\geq\frac{1}{a},\quad 5)\frac{2}{\gamma^j}\geq\frac{1}{\varepsilon^i},\quad 6)\frac{2}{\gamma^j}\geq\frac{1}{\delta^k}$

αφού πρόκειται για τις περιπτώσεις όπου οι χρήστες 1 και 2 έχουν επιλέξει ίδια σύνδεση, με χωρητικότητα μεγαλύτερη από αυτήν της σύνδεσης που επέλεξε ο χρήστης 3.

1) <u>Απόδειξη:</u> Από το c) έχουμε τη σχέση $\frac{1}{a}\leq\frac{1+p_1^t+p_2^t}{\zeta^t}$ Υποθέτουμε ότι $\frac{2}{\zeta^t}\geq\frac{1}{a}$, εφόσον $1<1+p_1^t+p_2^t<3$.
2) <u>Απόδειξη:</u> Από το a) έχουμε τη σχέση $\frac{1+p_2^t}{\zeta^t}=\frac{1+p_3^i}{\varepsilon^i}$ από την οποία προκύπτει $\frac{2}{\zeta^t}\geq\frac{1}{\varepsilon^i}$,
3) <u>Απόδειξη:</u> Από το b) έχουμε τη σχέση $\frac{1+p_3^k}{\delta^k}=\frac{1+p_1^t}{\zeta^t}$ από την οποία προκύπτει $\frac{2}{\zeta^t}\geq\frac{1}{\delta^k}$,
4) <u>Απόδειξη:</u> Από το c) έχουμε τη σχέση $\frac{1+p_1^j+p_2^j}{\gamma^j}=\frac{1}{a}$. Υποθέτουμε ότι $\frac{2}{\gamma^j}\geq\frac{1}{a}$, εφόσον $1<1+p_1^j+p_2^j<3$.
5) <u>Απόδειξη:</u> Από το c) έχουμε τη σχέση $\frac{1+p_1^i}{\varepsilon^i}=\frac{1+p_1^j+p_2^j}{\gamma^j}$. Υποθέτουμε ότι $\frac{2}{\gamma^j}\geq\frac{1}{\varepsilon^i}$, εφόσον $1<1+p_1^j+p_2^j<3$.
6) <u>Απόδειξη:</u> Από το c) έχουμε τη σχέση $\frac{1+p_2^k}{\delta^k}=\frac{1+p_1^j+p_2^j}{\gamma^j}$. Υποθέτουμε ότι $\frac{2}{\gamma^j}\geq\frac{1}{\varepsilon^i}$, εφόσον $1<1+p_1^j+p_2^j<3$.

Για την απόδειξη του ii) θα χρειαστεί να αποδείξουμε μόνο την ακόλουθη περίπτωση

1) $\frac{2}{\zeta^t} \geq \frac{1}{\gamma^j}$

αφού πρόκειται για την περίπτωση όπου οι χρήστες 1 και 2 έχουν επιλέξει ίδια σύνδεση, με χωρητικότητα μικρότερη από αυτήν της σύνδεσης που επέλεξε ο χρήστης 3.

1) <u>Απόδειξη:</u> Από το a) έχουμε τη σχέση $\frac{1+p_2^t}{\zeta^t} = \frac{1+p_2^j+p_3^j}{\gamma^j}$ από την οποία προκύπτει $\frac{2}{\zeta^t} \geq \frac{1}{\gamma^j}$.

<u>Απόδειξη (2)</u>

Για την απόδειξη του i) θα χρειαστεί να αποδείξουμε τις ακόλουθες περιπτώσεις

1) $\frac{2}{\varepsilon^i} \geq \frac{1}{\beta}$, 2) $\frac{2}{\varepsilon^i} \geq \frac{1}{\delta^k}$, 3) $\frac{2}{\gamma^j} \geq \frac{1}{\beta}$, 4) $\frac{2}{\gamma^j} \geq \frac{1}{\delta^k}$, 5) $\frac{2}{\gamma^j} \geq \frac{1}{\zeta^t}$

αφού πρόκειται για τις περιπτώσεις όπου οι χρήστες 1 και 3 έχουν επιλέξει ίδια σύνδεση, με χωρητικότητα μεγαλύτερη από αυτήν της σύνδεσης που επέλεξε ο χρήστης 2.

1) <u>Απόδειξη:</u> Από το c) έχουμε τη σχέση $\frac{1+p_1^i}{\varepsilon^i} = \frac{1}{a}$ από την οποία προκύπτει $\frac{2}{\varepsilon^i} \geq \frac{1}{a}$. Επίσης ισχύει ότι $\frac{1}{a} \geq \frac{1}{\beta}$, άρα $\frac{2}{\varepsilon^i} \geq \frac{1}{\beta}$.
2) <u>Απόδειξη:</u> Από το c) έχουμε τη σχέση $\frac{1+p_2^k}{\delta^k} = \frac{1+p_1^i}{\varepsilon^i}$ από την οποία προκύπτει $\frac{2}{\varepsilon^i} \geq \frac{1}{\delta^k}$.
3) <u>Απόδειξη:</u> Από το b) έχουμε ότι $\frac{1+p_1^j+p_3^j}{\gamma^j} = \frac{1}{\beta}$. Υποθέτουμε ότι $\frac{2}{\gamma^j} \geq \frac{1}{\beta}$, εφόσον $1 < 1 + p_1^j + p_3^j < 3$.
4) <u>Απόδειξη:</u> Από την απόδειξη 1-5 έχουμε ότι ισχύει $\frac{2}{\gamma^j} \geq \frac{1}{\varepsilon^i}$ ενώ ισχύει επίσης $\frac{1}{\varepsilon^i} \geq \frac{1}{\delta^k}$, επομένως προκύπτει ότι $\frac{2}{\gamma^j} \geq \frac{1}{\delta^k}$.
5) <u>Απόδειξη:</u> Από το b) έχουμε τη σχέση $\frac{1+p_1^t}{\zeta^t} = \frac{1+p_1^j+p_3^j}{\gamma^j}$ από την οποία προκύπτει $\frac{2}{\gamma^j} \geq \frac{1}{\zeta^t}$.

Για την απόδειξη του ii) θα χρειαστεί να αποδείξουμε μόνο την ακόλουθη περίπτωση

1) $\frac{2}{\varepsilon^i} \geq \frac{1}{\gamma^j}$

αφού πρόκειται για την περίπτωση όπου οι χρήστες 1 και 3 έχουν επιλέξει ίδια σύνδεση, με χωρητικότητα μικρότερη από αυτήν της σύνδεσης που επέλεξε ο χρήστης 2.

1) <u>Απόδειξη:</u> Από το c) έχουμε τη σχέση $\frac{1+p_1^i}{\varepsilon^i} = \frac{1+p_1^j+p_2^j}{\gamma^j}$ από την οποία προκύπτει $\frac{2}{\varepsilon^i} \geq \frac{1}{\gamma^j}$.

<u>Απόδειξη (3)</u>

Για την απόδειξη του i) θα χρειαστεί να αποδείξουμε τις ακόλουθες περιπτώσεις

1) $\frac{2}{\delta^k} \geq \frac{1}{\gamma}$, 2) $\frac{2}{\gamma^j} \geq \frac{1}{\gamma}$, 3) $\frac{2}{\gamma^j} \geq \frac{1}{\varepsilon^i}$, 4) $\frac{2}{\gamma^j} \geq \frac{1}{\zeta^t}$

αφού πρόκειται για τις περιπτώσεις όπου οι χρήστες 2 και 3 έχουν επιλέξει ίδια σύνδεση, με χωρητικότητα μεγαλύτερη από αυτήν της σύνδεσης που επέλεξε ο χρήστης 1.

1) Από το b) έχουμε τη σχέση $\frac{1+p_3^k}{\delta^k} \geq \frac{1}{\beta}$ ενώ ισχύει επίσης $\frac{1}{\beta} \geq \frac{1}{\gamma}$, επομένως προκύπτει ότι $\frac{2}{\delta^k} \geq \frac{1}{\gamma}$.
2) Από το a) έχουμε τη σχέση $\frac{1+p_2^j+p_3^j}{\gamma^j} = \frac{1}{\gamma}$. Υποθέτουμε ότι $\frac{2}{\gamma^j} \geq \frac{1}{\gamma}$, εφόσον $1 < 1 + p_2^j + p_3^j < 3$.
3) Από το a) έχουμε τη σχέση $\frac{1+p_3^i}{\varepsilon^i} = \frac{1+p_2^j+p_3^j}{\gamma^j}$. Υποθέτουμε ότι $\frac{2}{\gamma^j} \geq \frac{1}{\varepsilon^i}$, εφόσον $1 < 1 + p_2^j + p_3^j < 3$.
4) Από το b) έχουμε τη σχέση $\frac{1+p_1^t}{\zeta^t} = \frac{1+p_1^j+p_3^j}{\gamma^j}$. Υποθέτουμε ότι $\frac{2}{\gamma^j} \geq \frac{1}{\zeta^t}$, εφόσον $1 < 1 + p_1^j + p_3^j < 3$.

Για την απόδειξη του ii) θα χρειαστεί να αποδείξουμε τις ακόλουθες περιπτώσεις

1) $\frac{2}{\delta^k} \geq \frac{1}{\gamma^j}$, 2) $\frac{2}{\delta^k} \geq \frac{1}{\varepsilon^i}$, 3) $\frac{2}{\delta^k} \geq \frac{1}{\zeta^t}$

αφού πρόκειται για την περίπτωση όπου οι χρήστες 2 και 3 έχουν επιλέξει ίδια σύνδεση, με χωρητικότητα μικρότερη από αυτήν της σύνδεσης που επέλεξε ο χρήστης 1.

1) Σύμφωνα με την Πρόταση 3.3 έχουμε ότι $\frac{1}{\delta^k} \geq \frac{1}{\gamma^j}$ άρα προκύπτει $\frac{2}{\delta^k} \geq \frac{1}{\gamma^j}$

2) Σύμφωνα με την Πρόταση 3.3 έχουμε ότι $\frac{1}{\delta^k} \geq \frac{1}{\varepsilon^i}$ άρα προκύπτει $\frac{2}{\delta^k} \geq \frac{1}{\varepsilon^i}$

3) Σύμφωνα με την Πρόταση 3.3 έχουμε ότι $\frac{1}{\delta^k} \geq \frac{1}{\zeta^t}$ άρα προκύπτει $\frac{2}{\delta^k} \geq \frac{1}{\zeta^t}$

Κεφάλαιο 4

Το Κοινωνικό Κόστος μιας Μικτής Ανάθεσης

4.1 Παρουσίαση Λήμματος

Θα αποδείξουμε το παρακάτω βοηθητικό αξίωμα :

Λήμμα 4.1 *Το κοινωνικό κόστος μιας αυθαίρετης μικτής ανάθεσης* ***P*** *ικανοποιεί*

$$
\begin{aligned}
SC(P) - IC_3(P) \\
&= \sum_{1\le k<j<i\le m} p_3^k\, p_2^i\, p_1^j \left(\frac{1}{c^i} - \frac{1}{c^k}\right) + \sum_{1\le j<k<i\le m} p_3^k\, p_2^i\, p_1^j \left(\frac{1}{c^i} - \frac{1}{c^k}\right) \\
&+ \sum_{1\le k<i<j\le m} p_3^k\, p_2^i\, p_1^j \left(\frac{1}{c^j} - \frac{1}{c^k}\right) + \sum_{1\le i<k<j\le m} p_3^k\, p_2^i\, p_1^j \left(\frac{1}{c^j} - \frac{1}{c^k}\right) \\
&+ \sum_{1\le i<k\le m} p_3^k\, p_2^i\, p_1^j \left(\frac{2}{c^i} - \frac{1}{c^k}\right) + \sum_{1\le k<i\le m} p_3^k\, p_2^i\, p_1^j \left(\frac{2}{c^i} - \frac{1}{c^k}\right)
\end{aligned}
$$

4.2 Απόδειξη

Από τον ορισμό του Κοινωνικού Κόστους,

$$SC(P) = \sum_{1\le i,j,k\le m} p_3^k\, p_2^i\, p_1^j \cdot \begin{cases} \dfrac{1}{min\{c^i, c^j, c^k\}}, i \neq j, i \neq k, j \neq k \\ max\left\{\frac{2}{c^i}, \frac{1}{c^k}\right\}, i = j, i \neq k \\ max\left\{\frac{2}{c^k}, \frac{1}{c^j}\right\}, i = k, j \neq k \\ max\left\{\frac{2}{c^j}, \frac{1}{c^i}\right\}, j = k, i \neq j \\ \dfrac{3}{c^i}, i = j = k \end{cases}$$

Το οποίο μας δίνει σύμφωνα με την Πρόταση 3.4:

$$= \sum_{1\le i,j,k\le m} p_3^k\, p_2^i\, p_1^j \cdot \begin{cases} \dfrac{1}{c^i}, i > j > k \\ \dfrac{1}{c^i}, i > k > j \\ \dfrac{1}{c^j}, j > i > k \\ \dfrac{1}{c^j}, j > k > i \\ \dfrac{1}{c^k}, k > i > j \\ \dfrac{1}{c^k}, k > j > i \\ \dfrac{2}{c^i}, i = j, i > k \\ \dfrac{2}{c^i}, i = j, i < k \\ \dfrac{2}{c^k}, i = k, j < k \\ \dfrac{2}{c^j}, i = k, j > k \\ \dfrac{2}{c^j}, j = k, j > k \\ \dfrac{2}{c^j}, j = k, j > k \\ \dfrac{3}{c^i}, i = j = k \end{cases}$$

$$= \sum_{1\le k<j<i\le m} \frac{p_3^k\, p_2^i\, p_1^j}{c^i} + \sum_{1\le j<k<i\le m} \frac{p_3^k\, p_2^i\, p_1^j}{c^i} + \sum_{1\le k<i<j\le m} \frac{p_3^k\, p_2^i\, p_1^j}{c^j} + \sum_{1\le i<k<j\le m} \frac{p_3^k\, p_2^i\, p_1^j}{c^j}$$
$$+ \sum_{1\le j<i<k\le m} \frac{p_3^k\, p_2^i\, p_1^j}{c^k} + \sum_{1\le i<j<k\le m} \frac{p_3^k\, p_2^i\, p_1^j}{c^k} + \sum_{1\le k<i\le m} \frac{2p_3^k\, p_2^i\, p_1^j}{c^i}$$
$$+ \sum_{1\le i<k\le m} \frac{2p_3^k\, p_2^i\, p_1^j}{c^i} + \sum_{1\le i\le m} \frac{3p_3^k\, p_2^i\, p_1^j}{c^i}$$

Με τον ίδιο τρόπο έχουμε ότι το Αναμενόμενο Ατομικό Κόστος για το χρήστη 3,

$$IC_3(P) = \sum_{1\le k<j<i\le m} \frac{p_3^k\, p_2^i\, p_1^j}{c^k} + \sum_{1\le j<k<i\le m} \frac{p_3^k\, p_2^i\, p_1^j}{c^k} + \sum_{1\le k<i<j\le m} \frac{p_3^k\, p_2^i\, p_1^j}{c^k}$$
$$+ \sum_{1\le i<k<j\le m} \frac{p_3^k\, p_2^i\, p_1^j}{c^k} + \sum_{1\le j<i<k\le m} \frac{p_3^k\, p_2^i\, p_1^j}{c^k} + \sum_{1\le i<j<k\le m} \frac{p_3^k\, p_2^i\, p_1^j}{c^k}$$
$$+ \sum_{1\le k<i\le m} \frac{2p_3^k\, p_2^i\, p_1^j}{c^k} + \sum_{1\le i<k\le m} \frac{2p_3^k\, p_2^i\, p_1^j}{c^k} + \sum_{1\le k\le m} \frac{3p_3^k\, p_2^i\, p_1^j}{c^k}$$

Η αφαίρεση μας δίνει

$$SC(P) - IC_3(P)$$
$$= \sum_{1\le k<j<i\le m} p_3^k\, p_2^i\, p_1^j \left(\frac{1}{c^i} - \frac{1}{c^k}\right) + \sum_{1\le j<k<i\le m} p_3^k\, p_2^i\, p_1^j \left(\frac{1}{c^i} - \frac{1}{c^k}\right)$$
$$+ \sum_{1\le k<i<j\le m} p_3^k\, p_2^i\, p_1^j \left(\frac{1}{c^j} - \frac{1}{c^k}\right) + \sum_{1\le i<k<j\le m} p_3^k\, p_2^i\, p_1^j \left(\frac{1}{c^j} - \frac{1}{c^k}\right)$$
$$+ \sum_{1\le k<i\le m} p_3^k\, p_2^i\, p_1^j \left(\frac{2}{c^i} - \frac{1}{c^k}\right) + \sum_{1\le i<k\le m} p_3^k\, p_2^i\, p_1^j \left(\frac{2}{c^i} - \frac{1}{c^k}\right)$$

όπως χρειάζεται.

Κεφάλαιο 5

Το Κοινωνικό Κόστος της Πλήρως Μικτής Ισορροπίας Nash

5.1 Εφαρμογή Λήμματος 4.1

Εφαρμόζουμε το Λήμμα 4.1 με **F** για **P** το οποίο μας δίνει

$$SC(F) - IC_3^1(F) = SC(F) - \frac{1+2f^1}{\gamma^1} =$$

$$= \sum_{1\le k<j<i\le m} f^k\, f^i\, f^j \left(\frac{1}{c^i} - \frac{1}{c^k}\right) + \sum_{1\le j<k<i\le m} f^k\, f^i\, f^j \left(\frac{1}{c^i} - \frac{1}{c^k}\right)$$
$$+ \sum_{1\le k<i<j\le m} f^k\, f^i\, f^j \left(\frac{1}{c^j} - \frac{1}{c^k}\right) + \sum_{1\le i<k<j\le m} f^k\, f^i\, f^j \left(\frac{1}{c^j} - \frac{1}{c^k}\right)$$
$$+ \sum_{1\le k<i\le m} f^k\, f^i\, f^j \left(\frac{2}{c^i} - \frac{1}{c^k}\right) + \sum_{1\le i<k\le m} f^k\, f^i\, f^j \left(\frac{2}{c^i} - \frac{1}{c^k}\right).$$

5.2 Ανάλυση Όρων

Στη συνέχεια θα πρέπει να αναλύσουμε τον κάθε όρο της πιο πάνω διαφοράς χρησιμοποιώντας τους βοηθητικούς πίνακες στο Παράρτημα A. Ο πρώτος όρος σύμφωνα με τον αντίστοιχο πίνακα στο Παράρτημα A-1 αναλύεται ως εξής :

$$\sum_{1\le k<j<i\le m} f^k\, f^i\, f^j \left(\frac{1}{c^i}-\frac{1}{c^k}\right) =$$

$$\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le m_1} f^{r+j} \left(\frac{1}{\beta}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le m_1} f^{r+j} \left(\frac{1}{\beta}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le m_1} f^{r+j} \left(\frac{1}{\beta}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le r} f^k \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le m_1} f^{r+j} \left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+j} \left(\frac{1}{\beta}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+j} \left(\frac{1}{\beta}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+j} \left(\frac{1}{\beta}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le r} f^k \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+j} \left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j} \left(\frac{1}{\beta}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j} \left(\frac{1}{\beta}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le r} f^k \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j} \left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j} \left(\frac{1}{\beta}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le r} f^k \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j} \left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le r} f^{k} \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le r} f^{j}\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j}\left(\frac{1}{\zeta^i}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le r} f^{k} \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j}\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le r} f^{k} \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\le j\le r} f^{j}\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+j}\left(\frac{1}{\delta^i}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+j}\left(\frac{1}{\delta^i}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+j}\left(\frac{1}{\delta^i}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le r} f^{k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+j}\left(\frac{1}{\delta^i}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\delta^i}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\delta^i}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le r} f^{k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\delta^i}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j}\left(\frac{1}{\delta^i}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le r} f^{k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j}\left(\frac{1}{\delta^i}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le r} f^{k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le r} f^{j}\left(\frac{1}{\delta^i}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\varepsilon^i}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\varepsilon^i}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le r} f^{k} \sum_{1\le i\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\varepsilon^i}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j}\left(\frac{1}{\varepsilon^i}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le r} f^{k} \sum_{1\le i\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j}\left(\frac{1}{\varepsilon^i}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le r} f^{k} \sum_{1\le i\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+i} \sum_{1\le j\le r} f^{j}\left(\frac{1}{\varepsilon^i}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le m_1} f^{r+k} \sum_{1\le i\le m_3} f^{r+m_1+m_2+i} \sum_{1\le j\le m_2} f^{r+m_1+j}\left(\frac{1}{\alpha}-\frac{1}{\gamma}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{23}+k} \sum_{1\le i\le m_3} f^{r+m_1+m_2+i} \sum_{1\le j\le m_2} f^{r+m_1+j}\left(\frac{1}{\alpha}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_3} f^{r+m_1+m_2+i} \sum_{1\le j\le m_2} f^{r+m_1+j}\left(\frac{1}{\alpha}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_3} f^{r+m_1+m_2+i} \sum_{1\le j\le m_2} f^{r+m_1+j}\left(\frac{1}{\alpha}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le r} f^{k} \sum_{1\le i\le m_3} f^{r+m_1+m_2+i} \sum_{1\le j\le m_2} f^{r+m_1+j}\left(\frac{1}{\alpha}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{23}+k} \sum_{1\le i\le m_3} f^{r+m_1+m_2+i} \sum_{1\le j\le m_1} f^{r+j}\left(\frac{1}{\alpha}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_3} f^{r+m_1+m_2+i} \sum_{1\le j\le m_1} f^{r+j}\left(\frac{1}{\alpha}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_3} f^{r+m_1+m_2+i} \sum_{1\le j\le m_1} f^{r+j}\left(\frac{1}{\alpha}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le r} f^{k} \sum_{1\le i\le m_3} f^{r+m_1+m_2+i} \sum_{1\le j\le m_1} f^{r+j} \left(\frac{1}{\alpha}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_3} f^{r+m_1+m_2+i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j} \left(\frac{1}{\alpha}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le r} f^{k} \sum_{1\le i\le m_3} f^{r+m_1+m_2+i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j} \left(\frac{1}{\alpha}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_3} f^{r+m_1+m_2+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j} \left(\frac{1}{\alpha}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_3} f^{r+m_1+m_2+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j} \left(\frac{1}{\alpha}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le r} f^{k} \sum_{1\le i\le m_3} f^{r+m_1+m_2+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j} \left(\frac{1}{\alpha}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_3} f^{r+m_1+m_2+i} \sum_{1\le j\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+j} \left(\frac{1}{\alpha}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_3} f^{r+m_1+m_2+i} \sum_{1\le j\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+j} \left(\frac{1}{\alpha}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_3} f^{r+m_1+m_2+i} \sum_{1\le j\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+j} \left(\frac{1}{\alpha}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le r} f^{k} \sum_{1\le i\le m_3} f^{r+m_1+m_2+i} \sum_{1\le j\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+j} \left(\frac{1}{\alpha}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le r} f^{k} \sum_{1\le i\le m_3} f^{r+m_1+m_2+i} \sum_{1\le j\le r} f^{j} \left(\frac{1}{\alpha}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_1} f^{r+i} \sum_{1\le j\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+j} \left(\frac{1}{\gamma}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_1} f^{r+i} \sum_{1\le j\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+j} \left(\frac{1}{\gamma}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_1} f^{r+i} \sum_{1\le j\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+j} \left(\frac{1}{\gamma}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le r} f^{k} \sum_{1\le i\le m_1} f^{r+i} \sum_{1\le j\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+j}\left(\frac{1}{\gamma}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_1} f^{r+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\gamma}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_1} f^{r+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\gamma}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le r} f^{k} \sum_{1\le i\le m_1} f^{r+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\gamma}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_1} f^{r+i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j}\left(\frac{1}{\gamma}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le r} f^{k} \sum_{1\le i\le m_1} f^{r+i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j}\left(\frac{1}{\gamma}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le r} f^{k} \sum_{1\le i\le m_1} f^{r+i} \sum_{1\le j\le r} f^{j}\left(\frac{1}{\gamma}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le r} f^{k} \sum_{1\le i\le r} f^{i} \sum_{1\le j\le r} f^{j}\left(\frac{1}{\gamma^i}-\frac{1}{\gamma^k}\right)$$

Ο δεύτερος όρος σύμφωνα με τον βοηθητικό πίνακα Α-2 αναλύεται ως εξής :

$$\sum_{1\le j<k<i\le m} f^{k} f^{i} f^{j}\left(\frac{1}{c^i}-\frac{1}{c^k}\right)=$$

$$\sum_{1\le k\le m_1} f^{r+k} \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+j}\left(\frac{1}{\beta}-\frac{1}{\gamma}\right)$$

$$+\sum_{1\le k\le m_1} f^{r+k} \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\beta}-\frac{1}{\gamma}\right)$$

$$+\sum_{1\le k\le m_1} f^{r+k} \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j}\left(\frac{1}{\beta}-\frac{1}{\gamma}\right)$$

$$+\sum_{1\le k\le m_1} f^{r+k} \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le r} f^{j}\left(\frac{1}{\beta}-\frac{1}{\gamma}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+j}\left(\frac{1}{\beta}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\beta}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j}\left(\frac{1}{\beta}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le r} f^{j}\left(\frac{1}{\beta}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\beta}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j}\left(\frac{1}{\beta}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le r} f^{j}\left(\frac{1}{\beta}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j}\left(\frac{1}{\beta}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le r} f^{j}\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le r} f^{k} \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le r} f^{j}\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j}\left(\frac{1}{\zeta^i}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\le j\le r} f^{j}\left(\frac{1}{\zeta^i}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le r} f^{k} \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\le j\le r} f^{j}\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+j}\left(\frac{1}{\delta^i}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\delta^i}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j}\left(\frac{1}{\delta^i}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le r} f^{j}\left(\frac{1}{\delta^i}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\delta^i}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j}\left(\frac{1}{\delta^i}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le r} f^{j}\left(\frac{1}{\delta^i}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j}\left(\frac{1}{\delta^i}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le r} f^{j}\left(\frac{1}{\delta^i}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le r} f^{k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le r} f^{j}\left(\frac{1}{\delta^i}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\varepsilon^i}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j}\left(\frac{1}{\varepsilon^i}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+i} \sum_{1\le j\le r} f^{j}\left(\frac{1}{\varepsilon^i}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j}\left(\frac{1}{\varepsilon^i}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+i} \sum_{1\le j\le r} f^{j}\left(\frac{1}{\varepsilon^i}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le r} f^{k} \sum_{1\le i\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+i} \sum_{1\le j\le r} f^{j}\left(\frac{1}{\varepsilon^{i}}-\frac{1}{\gamma^{k}}\right)$$

$$+\sum_{1\le k\le m_2} f^{r+m_1+k} \sum_{1\le i\le m_3} f^{r+m_1+m_2+i} \sum_{1\le j\le m_1} f^{r+j}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)$$

$$+\sum_{1\le k\le m_2} f^{r+m_1+k} \sum_{1\le i\le m_3} f^{r+m_1+m_2+i} \sum_{1\le j\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{23}+j}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)$$

$$+\sum_{1\le k\le m_2} f^{r+m_1+k} \sum_{1\le i\le m_3} f^{r+m_1+m_2+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)$$

$$+\sum_{1\le k\le m_2} f^{r+m_1+k} \sum_{1\le i\le m_3} f^{r+m_1+m_2+i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)$$

$$+\sum_{1\le k\le m_2} f^{r+m_1+k} \sum_{1\le i\le m_3} f^{r+m_1+m_2+i} \sum_{1\le j\le r} f^{j}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)$$

$$+\sum_{1\le k\le m_1} f^{r+k} \sum_{1\le i\le m_3} f^{r+m_1+m_2+i} \sum_{1\le j\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{23}+j}\left(\frac{1}{\alpha}-\frac{1}{\gamma}\right)$$

$$+\sum_{1\le k\le m_1} f^{r+k} \sum_{1\le i\le m_3} f^{r+m_1+m_2+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\alpha}-\frac{1}{\gamma}\right)$$

$$+\sum_{1\le k\le m_1} f^{r+k} \sum_{1\le i\le m_3} f^{r+m_1+m_2+i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j}\left(\frac{1}{\alpha}-\frac{1}{\gamma}\right)$$

$$+\sum_{1\le k\le m_1} f^{r+k} \sum_{1\le i\le m_3} f^{r+m_1+m_2+i} \sum_{1\le j\le r} f^{j}\left(\frac{1}{\alpha}-\frac{1}{\gamma}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_3} f^{r+m_1+m_2+i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j}\left(\frac{1}{\alpha}-\frac{1}{\zeta^{k}}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_3} f^{r+m_1+m_2+i} \sum_{1\le j\le r} f^{j}\left(\frac{1}{\alpha}-\frac{1}{\zeta^{k}}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_3} f^{r+m_1+m_2+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\alpha}-\frac{1}{\varepsilon^{k}}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_3} f^{r+m_1+m_2+i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j}\left(\frac{1}{\alpha}-\frac{1}{\varepsilon^{k}}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_3} f^{r+m_1+m_2+i} \sum_{1\le j\le r} f^j \left(\frac{1}{\alpha}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_3} f^{r+m_1+m_2+i} \sum_{1\le j\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+j} \left(\frac{1}{\alpha}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_3} f^{r+m_1+m_2+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j} \left(\frac{1}{\alpha}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_3} f^{r+m_1+m_2+i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j} \left(\frac{1}{\alpha}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_3} f^{r+m_1+m_2+i} \sum_{1\le j\le r} f^j \left(\frac{1}{\alpha}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le r} f^k \sum_{1\le i\le m_3} f^{r+m_1+m_2+i} \sum_{1\le j\le r} f^j \left(\frac{1}{\alpha}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_1} f^{r+i} \sum_{1\le j\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+j} \left(\frac{1}{\gamma}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_1} f^{r+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j} \left(\frac{1}{\gamma}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_1} f^{r+i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j} \left(\frac{1}{\gamma}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_1} f^{r+i} \sum_{1\le j\le r} f^j \left(\frac{1}{\gamma}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_1} f^{r+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j} \left(\frac{1}{\gamma}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_1} f^{r+i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j} \left(\frac{1}{\gamma}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_1} f^{r+i} \sum_{1\le j\le r} f^j \left(\frac{1}{\gamma}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_1} f^{r+i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j} \left(\frac{1}{\gamma}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\leq k\leq m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\leq i\leq m_1} f^{r+i} \sum_{1\leq j\leq r} f^j \left(\frac{1}{\gamma}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\leq k\leq r} f^k \sum_{1\leq i\leq m_1} f^{r+i} \sum_{1\leq j\leq r} f^j \left(\frac{1}{\gamma}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\leq k\leq r} f^k \sum_{1\leq i\leq r} f^i \sum_{1\leq j\leq r} f^j \left(\frac{1}{\gamma^i}-\frac{1}{\gamma^k}\right)$$

Ο τρίτος όρος σύμφωνα με τον Πίνακα Α-3 αναλύεται ως εξής :

$$\sum_{1\leq k<i<j\leq m} f^k f^i f^j \left(\frac{1}{c^j}-\frac{1}{c^k}\right) =$$

$$\sum_{1\leq k\leq m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\leq i\leq m_1} f^{r+i} \sum_{1\leq j\leq m_2} f^{r+m_1+j} \left(\frac{1}{\beta}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\leq k\leq m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\leq i\leq m_1} f^{r+i} \sum_{1\leq j\leq m_2} f^{r+m_1+j} \left(\frac{1}{\beta}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\leq k\leq m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\leq i\leq m_1} f^{r+i} \sum_{1\leq j\leq m_2} f^{r+m_1+j} \left(\frac{1}{\beta}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\leq k\leq r} f^k \sum_{1\leq i\leq m_1} f^{r+i} \sum_{1\leq j\leq m_2} f^{r+m_1+j} \left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\leq k\leq m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\leq i\leq m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\leq j\leq m_2} f^{r+m_1+j} \left(\frac{1}{\beta}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\leq k\leq m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\leq i\leq m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\leq j\leq m_2} f^{r+m_1+j} \left(\frac{1}{\beta}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\leq k\leq m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\leq i\leq m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\leq j\leq m_2} f^{r+m_1+j} \left(\frac{1}{\beta}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\leq k\leq r} f^k \sum_{1\leq i\leq m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\leq j\leq m_2} f^{r+m_1+j} \left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\leq k\leq m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\leq i\leq m_{13}} f^{r+m_1+m_2+m_3+m_{12}+i} \sum_{1\leq j\leq m_2} f^{r+m_1+j} \left(\frac{1}{\beta}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+i} \sum_{1\le j\le m_2} f^{r+m_1+j} \left(\frac{1}{\beta}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le r} f^{k} \sum_{1\le i\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+i} \sum_{1\le j\le m_2} f^{r+m_1+j} \left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_2} f^{r+m_1+j} \left(\frac{1}{\beta}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le r} f^{k} \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_2} f^{r+m_1+j} \left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le r} f^{k} \sum_{1\le i\le r} f^{i} \sum_{1\le j\le m_2} f^{r+m_1+j} \left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j} \left(\frac{1}{\zeta^j}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le r} f^{k} \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j} \left(\frac{1}{\zeta^j}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le r} f^{k} \sum_{1\le i\le r} f^{i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j} \left(\frac{1}{\zeta^j}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+j} \left(\frac{1}{\delta^j}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+j} \left(\frac{1}{\delta^j}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+j} \left(\frac{1}{\delta^j}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le r} f^{k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+j} \left(\frac{1}{\delta^j}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+i} \sum_{1\le j\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+j} \left(\frac{1}{\delta^j}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+i} \sum_{1\le j\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+j} \left(\frac{1}{\delta^j}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le r} f^{k} \sum_{1\le i\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+i} \sum_{1\le j\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+j}\left(\frac{1}{\delta^j}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+j}\left(\frac{1}{\delta^j}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le r} f^{k} \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+j}\left(\frac{1}{\delta^j}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le r} f^{k} \sum_{1\le i\le r} f^{i} \sum_{1\le j\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+j}\left(\frac{1}{\delta^j}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\varepsilon^j}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\varepsilon^j}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le r} f^{k} \sum_{1\le i\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\varepsilon^j}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\varepsilon^j}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le r} f^{k} \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\varepsilon^j}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le r} f^{k} \sum_{1\le i\le r} f^{i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\varepsilon^j}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le m_1} f^{r+k} \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le m_3} f^{r+m_1+m_2+j}\left(\frac{1}{\alpha}-\frac{1}{\gamma}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{23}+k} \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le m_3} f^{r+m_1+m_2+j}\left(\frac{1}{\alpha}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le m_3} f^{r+m_1+m_2+j}\left(\frac{1}{\alpha}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le m_3} f^{r+m_1+m_2+j}\left(\frac{1}{\alpha}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le r} f^k \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le m_3} f^{r+m_1+m_2+j}\left(\frac{1}{\alpha}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{23}+k} \sum_{1\le i\le m_1} f^{r+i} \sum_{1\le j\le m_3} f^{r+m_1+m_2+j}\left(\frac{1}{\alpha}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_1} f^{r+i} \sum_{1\le j\le m_3} f^{r+m_1+m_2+j}\left(\frac{1}{\alpha}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_1} f^{r+i} \sum_{1\le j\le m_3} f^{r+m_1+m_2+j}\left(\frac{1}{\alpha}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le r} f^k \sum_{1\le i\le m_3} f^{r+i} \sum_{1\le j\le m_1} f^{r+m_1+m_2+j}\left(\frac{1}{\alpha}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_3} f^{r+m_1+m_2+j}\left(\frac{1}{\alpha}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le r} f^k \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_3} f^{r+m_1+m_2+j}\left(\frac{1}{\alpha}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+i} \sum_{1\le j\le m_3} f^{r+m_1+m_2+j}\left(\frac{1}{\alpha}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+i} \sum_{1\le j\le m_3} f^{r+m_1+m_2+j}\left(\frac{1}{\alpha}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le r} f^k \sum_{1\le i\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+i} \sum_{1\le j\le m_3} f^{r+m_1+m_2+j}\left(\frac{1}{\alpha}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_3} f^{r+m_1+m_2+j}\left(\frac{1}{\alpha}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_3} f^{r+m_1+m_2+j}\left(\frac{1}{\alpha}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_3} f^{r+m_1+m_2+j}\left(\frac{1}{\alpha}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le r} f^k \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_3} f^{r+m_1+m_2+j}\left(\frac{1}{\alpha}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le r} f^k \sum_{1\le i\le r} f^i \sum_{1\le j\le m_3} f^{r+m_1+m_2+j}\left(\frac{1}{\alpha}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_1} f^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_1} f^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_1} f^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le r} f^k \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_1} f^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+i} \sum_{1\le j\le m_1} f^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+i} \sum_{1\le j\le m_1} f^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le r} f^k \sum_{1\le i\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+i} \sum_{1\le j\le m_1} f^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_1} f^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le r} f^k \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_1} f^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le r} f^k \sum_{1\le i\le r} f^i \sum_{1\le j\le m_1} f^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le r} f^k \sum_{1\le i\le r} f^i \sum_{1\le j\le r} f^j\left(\frac{1}{\gamma^j}-\frac{1}{\gamma^k}\right)$$

Ο τέταρτος όρος σύμφωνα με τον Πίνακα Α-4 αναλύεται ως εξής :

$$\sum_{1\le i<k<j\le m} f^k f^i f^j\left(\frac{1}{c^j}-\frac{1}{c^k}\right)=$$

$$\sum_{1\le k\le m_1} f^{r+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_2} f^{r+m_1+j}\left(\frac{1}{\beta}-\frac{1}{\gamma}\right)$$

$$+\sum_{1\le k\le m_1} f^{r+k} \sum_{1\le i\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+i} \sum_{1\le j\le m_2} f^{r+m_1+j}\left(\frac{1}{\beta}-\frac{1}{\gamma}\right)$$

$$+\sum_{1\le k\le m_1} f^{r+k} \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_2} f^{r+m_1+j}\left(\frac{1}{\beta}-\frac{1}{\gamma}\right)$$

$$+\sum_{1\le k\le m_1} f^{r+k} \sum_{1\le i\le r} f^{i} \sum_{1\le j\le m_2} f^{r+m_1+j}\left(\frac{1}{\beta}-\frac{1}{\gamma}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_2} f^{r+m_1+j}\left(\frac{1}{\beta}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+i} \sum_{1\le j\le m_2} f^{r+m_1+j}\left(\frac{1}{\beta}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_2} f^{r+m_1+j}\left(\frac{1}{\beta}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le r} f^{i} \sum_{1\le j\le m_2} f^{r+m_1+j}\left(\frac{1}{\beta}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+i} \sum_{1\le j\le m_2} f^{r+m_1+j}\left(\frac{1}{\beta}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_2} f^{r+m_1+j}\left(\frac{1}{\beta}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le r} f^{i} \sum_{1\le j\le m_2} f^{r+m_1+j}\left(\frac{1}{\beta}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j}\left(\frac{1}{\beta}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le r} f^{i} \sum_{1\le j\le m_2} f^{r+m_1+j}\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le r} f^{k} \sum_{1\le i\le r} f^{i} \sum_{1\le j\le m_2} f^{r+m_1+j}\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j} \left(\frac{1}{\zeta^j}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le r} f^{i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j} \left(\frac{1}{\zeta^j}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le r} f^{k} \sum_{1\le i\le r} f^{i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j} \left(\frac{1}{\zeta^j}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+j} \left(\frac{1}{\delta^j}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+i} \sum_{1\le j\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+j} \left(\frac{1}{\delta^j}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+j} \left(\frac{1}{\delta^j}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le r} f^{i} \sum_{1\le j\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+j} \left(\frac{1}{\delta^j}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+i} \sum_{1\le j\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+j} \left(\frac{1}{\delta^j}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+j} \left(\frac{1}{\delta^j}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le r} f^{i} \sum_{1\le j\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+j} \left(\frac{1}{\delta^j}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+j} \left(\frac{1}{\delta^j}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le r} f^{i} \sum_{1\le j\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+j} \left(\frac{1}{\delta^j}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le r} f^{k} \sum_{1\le i\le r} f^{i} \sum_{1\le j\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+j} \left(\frac{1}{\delta^j}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j} \left(\frac{1}{\varepsilon^j}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\varepsilon^j}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le r} f^{i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\varepsilon^j}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\varepsilon^j}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le r} f^{i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\varepsilon^j}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le r} f^{k} \sum_{1\le i\le r} f^{i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\varepsilon^j}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le m_2} f^{r+m_1+k} \sum_{1\le i\le m_1} f^{r+i} \sum_{1\le j\le m_3} f^{r+m_1+m_2+j}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)$$

$$+\sum_{1\le k\le m_2} f^{r+m_1+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{23}+i} \sum_{1\le j\le m_3} f^{r+m_1+m_2+j}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)$$

$$+\sum_{1\le k\le m_2} f^{r+m_1+k} \sum_{1\le i\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+i} \sum_{1\le j\le m_3} f^{r+m_1+m_2+j}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)$$

$$+\sum_{1\le k\le m_2} f^{r+m_1+k} \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_3} f^{r+m_1+m_2+j}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)$$

$$+\sum_{1\le k\le m_2} f^{r+m_1+k} \sum_{1\le i\le r} f^{i} \sum_{1\le j\le m_3} f^{r+m_1+m_2+j}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)$$

$$+\sum_{1\le k\le m_1} f^{r+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{23}+i} \sum_{1\le j\le m_3} f^{r+m_1+m_2+j}\left(\frac{1}{\alpha}-\frac{1}{\gamma}\right)$$

$$+\sum_{1\le k\le m_1} f^{r+k} \sum_{1\le i\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+i} \sum_{1\le j\le m_3} f^{r+m_1+m_2+j}\left(\frac{1}{\alpha}-\frac{1}{\gamma}\right)$$

$$+\sum_{1\le k\le m_1} f^{r+k} \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_3} f^{r+m_1+m_2+j}\left(\frac{1}{\alpha}-\frac{1}{\gamma}\right)$$

$$+\sum_{1\le k\le m_1} f^{r+k} \sum_{1\le i\le r} f^{i} \sum_{1\le j\le m_3} f^{r+m_1+m_2+j}\left(\frac{1}{\alpha}-\frac{1}{\gamma}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_3} f^{r+m_1+m_2+j}\left(\frac{1}{\alpha}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le r} f^{i} \sum_{1\le j\le m_3} f^{r+m_1+m_2+j}\left(\frac{1}{\alpha}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+i} \sum_{1\le j\le m_3} f^{r+m_1+m_2+j}\left(\frac{1}{\alpha}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_3} f^{r+m_1+m_2+j}\left(\frac{1}{\alpha}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le r} f^{i} \sum_{1\le j\le m_3} f^{r+m_1+m_2+j}\left(\frac{1}{\alpha}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_3} f^{r+m_1+m_2+j}\left(\frac{1}{\alpha}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+i} \sum_{1\le j\le m_3} f^{r+m_1+m_2+j}\left(\frac{1}{\alpha}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_3} f^{r+m_1+m_2+j}\left(\frac{1}{\alpha}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le r} f^{i} \sum_{1\le j\le m_3} f^{r+m_1+m_2+j}\left(\frac{1}{\alpha}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le r} f^{k} \sum_{1\le i\le r} f^{i} \sum_{1\le j\le m_3} f^{r+m_1+m_2+j}\left(\frac{1}{\alpha}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_1} f^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+i} \sum_{1\le j\le m_1} f^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_1} f^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le r} f^{i} \sum_{1\le j\le m_1} f^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+i} \sum_{1\le j\le m_1} f^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_1} f^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le r} f^{i} \sum_{1\le j\le m_1} f^{jr+}\left(\frac{1}{\gamma}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_1} f^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le r} f^{i} \sum_{1\le j\le m_1} f^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le r} f^{k} \sum_{1\le i\le r} f^{i} \sum_{1\le j\le m_1} f^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le r} f^{k} \sum_{1\le i\le r} f^{i} \sum_{1\le j\le r} f^{j}\left(\frac{1}{\gamma^j}-\frac{1}{\gamma^k}\right)$$

Ο πέμπτος όρος σύμφωνα με τοπ Πίνακα Α-5 αναλύεται ως εξής :

$$\sum_{1\le k<i\le m} f^k\, f^i\, f^j\left(\frac{2}{c^i}-\frac{1}{c^k}\right)=$$

$$\sum_{1\le k\le m_3} f^{r+m_1+m_2+k} \sum_{1\le i\le m_2} f^{r+m_1+i} f^{r+m_1+i}\left(\frac{2}{\beta}-\frac{1}{a}\right)$$

$$+\sum_{1\le k\le m_3} f^{r+m_1+m_2+k} \sum_{1\le i\le m_1} f^{r+i} f^{r+i}\left(\frac{2}{\beta}-\frac{1}{a}\right)$$

$$+\sum_{1\le k\le m_3} f^{r+m_1+m_2+k} \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} f^{r+m_1+m_2+m_3+i}\left(\frac{2}{\zeta^i}-\frac{1}{a}\right)$$

$$+\sum_{1\le k\le m_3} f^{r+m_1+m_2+k} \sum_{1\le i\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+i} f^{r+m_1+m_2+m_3+m_{12}+i}\left(\frac{2}{\varepsilon^i}-\frac{1}{a}\right)$$

$$+\sum_{1\le k\le m_3} f^{r+m_1+m_2+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{23}+i} f^{r+m_1+m_2+m_3+m_{12}+m_{23}+i} \left(\frac{2}{\delta^i}-\frac{1}{a}\right)$$

$$+\sum_{1\le k\le m_3} f^{r+m_1+m_2+k} \sum_{1\le i\le r} f^i f^i \left(\frac{2}{\gamma^i}-\frac{1}{a}\right)$$

$$+\sum_{1\le k\le m_2} f^{r+m_1+k} \sum_{1\le i\le m_1} f^{r+i} f^{r+i} \left(\frac{2}{\beta}-\frac{1}{a}\right)$$

$$+\sum_{1\le k\le m_2} f^{r+m_1+k} \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} f^{r+m_1+m_2+m_3+i} \left(\frac{2}{\zeta^i}-\frac{1}{\beta}\right)$$

$$+\sum_{1\le k\le m_2} f^{r+m_1+k} \sum_{1\le i\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+i} f^{r+m_1+m_2+m_3+m_{12}+i} \left(\frac{2}{\varepsilon^i}-\frac{1}{\beta}\right)$$

$$+\sum_{1\le k\le m_2} f^{r+m_1+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{23}+i} f^{r+m_1+m_2+m_3+m_{12}+m_{23}+i} \left(\frac{2}{\delta^i}-\frac{1}{a}\right)$$

$$+\sum_{1\le k\le m_2} f^{r+m_1+k} \sum_{1\le i\le r} f^i f^i \left(\frac{2}{\gamma^i}-\frac{1}{a}\right)$$

$$+\sum_{1\le k\le m_1} f^{r+k} \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} f^{r+m_1+m_2+m_3+i} \left(\frac{2}{\zeta^i}-\frac{1}{\gamma}\right)$$

$$+\sum_{1\le k\le m_1} f^{r+k} \sum_{1\le i\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+i} f^{r+m_1+m_2+m_3+m_{12}+i} \left(\frac{2}{\varepsilon^i}-\frac{1}{\gamma}\right)$$

$$+\sum_{1\le k\le m_1} f^{r+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \left(\frac{2}{\delta^i}-\frac{1}{\gamma}\right)$$

$$+\sum_{1\le k\le m_1} f^{r+k} \sum_{1\le i\le r} f^i f^i \left(\frac{2}{\gamma^i}-\frac{1}{\gamma}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} f^{r+m_1+m_2+m_3+i} \left(\frac{2}{\zeta^i}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+i} f^{r+m_1+m_2+m_3+m_{12}+i} \left(\frac{2}{\varepsilon^i}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \left(\frac{2}{\delta^i}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\leq k\leq m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{23}+k} \sum_{1\leq i\leq r} f^i f^i \left(\frac{2}{\gamma^i}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\leq k\leq m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\leq i\leq m_{12}} f^{r+m_1+m_2+m_3+i} f^{r+m_1+m_2+m_3+i} \left(\frac{2}{\zeta^i}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\leq k\leq m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\leq i\leq m_{13}} f^{r+m_1+m_2+m_3+m_{12}+i} f^{r+m_1+m_2+m_3+m_{12}+i} \left(\frac{2}{\varepsilon^i}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\leq k\leq m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\leq i\leq r} f^i f^i \left(\frac{2}{\gamma^i}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\leq k\leq m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\leq i\leq m_{12}} f^{r+m_1+m_2+m_3+i} f^{r+m_1+m_2+m_3+i} \left(\frac{2}{\zeta^i}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\leq k\leq m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\leq i\leq r} f^i f^i \left(\frac{2}{\gamma^i}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\leq k\leq r} f^k \sum_{1\leq i\leq r} f^i f^i \left(\frac{2}{\gamma^i}-\frac{1}{\zeta^k}\right)$$

Ο έκτος όρος σύμφωνα με τον Πίνακα Α-6 αναλύεται ως εξής :

$$\sum_{1\leq i<k\leq m} f^k\, f^i\, f^j \left(\frac{2}{c^i}-\frac{1}{c^k}\right) =$$

$$\sum_{1\leq k\leq m_2} f^{r+m_1+k} \sum_{1\leq i\leq m_3} f^{r+m_1+m_2+i} f^{r+m_1+m_2+i} \left(\frac{2}{a}-\frac{1}{\beta}\right)$$

$$+\sum_{1\leq k\leq m_1} f^{r+k} \sum_{1\leq i\leq m_3} f^{r+m_1+m_2+i} f^{r+m_1+m_2+i} \left(\frac{2}{\alpha}-\frac{1}{\beta}\right)$$

$$+\sum_{1\leq k\leq m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\leq i\leq m_3} f^{r+m_1+m_2+i} f^{r+m_1+m_2+i} \left(\frac{2}{\alpha}-\frac{1}{\zeta^i}\right)$$

$$+\sum_{1\leq k\leq m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\leq i\leq m_3} f^{r+m_1+m_2+i} f^{r+m_1+m_2+i} \left(\frac{2}{\alpha}-\frac{1}{\varepsilon^i}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{23}+k} \sum_{1\le i\le m_3} f^{r+m_1+m_2+i} f^{r+m_1+m_2+i}\left(\frac{2}{\alpha}-\frac{1}{\delta^i}\right)$$

$$+\sum_{1\le k\le r} f^k \sum_{1\le i\le m_3} f^{r+m_1+m_2+i} f^{r+m_1+m_2+i}\left(\frac{2}{\alpha}-\frac{1}{\gamma^i}\right)$$

$$+\sum_{1\le k\le m_1} f^{r+k} \sum_{1\le i\le m_2} f^{r+m_1+i} f^{r+m_1+i}\left(\frac{2}{\alpha}-\frac{1}{\beta}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_2} f^{r+m_1+i} f^{r+m_1+i}\left(\frac{2}{\beta}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_2} f^{r+m_1+i} f^{r+m_1+i}\left(\frac{2}{\beta}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_2} f^{r+m_1+i} f^{r+m_1+i}\left(\frac{2}{\beta}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le r} f^k \sum_{1\le i\le m_2} f^{r+m_1+i} f^{r+m_1+i}\left(\frac{2}{\beta}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_1} f^{r+i} f^{r+i}\left(\frac{2}{\gamma}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_1} f^{r+i} f^{r+i}\left(\frac{2}{\gamma}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_1} f^{r+i} f^{r+i}\left(\frac{2}{\gamma}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le r} f^k \sum_{1\le i\le m_1} f^{r+i} f^{r+i}\left(\frac{2}{\gamma}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i}\left(\frac{2}{\delta^i}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i}\left(\frac{2}{\delta^i}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i}\left(\frac{2}{\delta^i}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le r} f^k \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{23}+i} f^{r+m_1+m_2+m_3+m_{12}+m_{23}+i}\left(\frac{2}{\delta^i}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+i} f^{r+m_1+m_2+m_3+m_{12}+i}\left(\frac{2}{\varepsilon^i}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+i} f^{r+m_1+m_2+m_3+m_{12}+i}\left(\frac{2}{\varepsilon^i}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le r} f^k \sum_{1\le i\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+i} f^{r+m_1+m_2+m_3+m_{12}+i}\left(\frac{2}{\varepsilon^i}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} f^{r+m_1+m_2+m_3+i}\left(\frac{2}{\zeta^i}-\frac{1}{\zeta^k}\right)$$

$$+\sum_{1\le k\le r} f^k \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} f^{r+m_1+m_2+m_3+i}\left(\frac{2}{\zeta^i}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le r} f^k \sum_{1\le i\le r} f^i f^i\left(\frac{2}{\zeta^i}-\frac{1}{\gamma^k}\right)$$

Το άθροισμα των 6 πιο πάνω όρων μας δίνει το Κοινωνικό Κόστος της Πλήρως Μικτής Ισορροπίας Nash σύμφωνα με το Λήμμα 4.1.

Κεφάλαιο 6

Το Κοινωνικό Κόστος της Αυθαίρετης Ισορροπίας Nash

6.1 Εφαρμογή Λήμματος 4.1

Εφαρμόζουμε το Λήμμα 4.1 οποίο μας δίνει

$$SC(P) - IC_3^1(P) = SC(P) - \frac{1 + p_1^1 + p_2^1}{\gamma^1} =$$

$$= \sum_{1 \le k < j < i \le m} p_3^k\, p_2^i\, p_1^j \left(\frac{1}{c^i} - \frac{1}{c^k}\right) + \sum_{1 \le j < k < i \le m} p_3^k\, p_2^i\, p_1^j \left(\frac{1}{c^i} - \frac{1}{c^k}\right)$$
$$+ \sum_{1 \le i < k < j \le m} p_3^k\, p_2^i\, p_1^j \left(\frac{1}{c^j} - \frac{1}{c^k}\right) + \sum_{1 \le k < i < j \le m} p_3^k\, p_2^i\, p_1^j \left(\frac{1}{c^j} - \frac{1}{c^k}\right)$$
$$+ \sum_{1 \le k < i \le m} p_3^k\, p_2^i\, p_1^j \left(\frac{2}{c^i} - \frac{1}{c^k}\right) + \sum_{1 \le i < k \le m} p_3^k\, p_2^i\, p_1^j \left(\frac{2}{c^i} - \frac{1}{c^k}\right).$$

5.1 Ανάλυση Όρων

Στη συνέχεια θα πρέπει να αναλύσουμε τον κάθε όρο της πιο πάνω διαφοράς χρησιμοποιώντας τους βοηθητικούς πίνακες στο Παράρτημα Β. Ο πρώτος όρος σύμφωνα με τον αντίστοιχο πίνακα Β-1 αναλύεται ως εξής :

$$\sum_{1\le k<j<i\le m} p_3^k\, p_2^i\, p_1^j \left(\frac{1}{c^i}-\frac{1}{c^k}\right) =$$

$$\sum_{1\le k\le m_{23}} p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_2} p_2^{r+m_1+i} \sum_{1\le j\le m_1} p_1^{r+j} \left(\frac{1}{\beta}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{13}} p_3^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_2} p_2^{r+m_1+i} \sum_{1\le j\le m_1} p_1^{r+j} \left(\frac{1}{\beta}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le r} p_3^k \sum_{1\le i\le m_2} p_2^{r+m_1+i} \sum_{1\le j\le m_1} p_1^{r+j} \left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le r} p_3^k \sum_{1\le i\le m_2} p_2^{r+m_1+i} \sum_{1\le j\le m_{12}} p_1^{r+m_1+m_2+m_3+j} \left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le r} p_3^k \sum_{1\le i\le m_2} p_2^{r+m_1+i} \sum_{1\le j\le m_{13}} p_1^{r+m_1+m_2+m_3+m_{12}+j} \left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le m_{13}} p_3^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_2} p_2^{r+m_1+i} \sum_{1\le j\le m_{13}} p_1^{r+m_1+m_2+m_3+m_{12}+j} \left(\frac{1}{\beta}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le r} p_3^k \sum_{1\le i\le m_2} p_2^{r+m_1+i} \sum_{1\le j\le r} p_1^j \left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le r} p_3^k \sum_{1\le i\le m_{12}} p_2^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_{12}} p_1^{r+m_1+m_2+m_3+j} \left(\frac{1}{\zeta^i}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le r} p_3^k \sum_{1\le i\le m_{12}} p_2^{r+m_1+m_2+m_3+i} \sum_{1\le j\le r} p_1^j \left(\frac{1}{\zeta^i}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le r} p_3^k \sum_{1\le i\le m_{23}} p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_{12}} p_1^j \left(\frac{1}{\delta^i}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le r} p_3^k \sum_{1\le i\le m_{23}} p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_{13}} p_1^{r+m_1+m_2+m_3+m_{12}+j} \left(\frac{1}{\delta^i}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le r} p_3^k \sum_{1\le i\le m_{23}} p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le r} p_1^j \left(\frac{1}{\delta^i}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le r} p_3^k \sum_{1\le i\le r} p_2^i \sum_{1\le j\le r} p_1^j \left(\frac{1}{\gamma^i}-\frac{1}{\gamma^k}\right)$$

Ο δεύτερος όρος σύμφωνα με τον Πίνακα Β-2 αναλύεται ως εξής :

$$\sum_{1\le j<k<i\le m} p_3^k\, p_2^i\, p_1^j \left(\frac{1}{c^i}-\frac{1}{c^k}\right)=$$

$$\sum_{1\le k\le m_{13}} p_3^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_2} p_2^i \sum_{1\le j\le m_{12}} p_1^{r+m_1+m_2+m_3+j}\left(\frac{1}{\beta}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{13}} p_3^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_2} p_2^{r+m_1+i} \sum_{1\le j\le m_{13}} p_1^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\beta}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{13}} p_3^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_2} p_2^{r+m_1+i} \sum_{1\le j\le r} p_1^j\left(\frac{1}{\beta}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{23}} p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_2} p_2^{r+m_1+i} \sum_{1\le j\le r} p_1^j\left(\frac{1}{\beta}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{23}} p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_2} p_2^{r+m_1+i} \sum_{1\le j\le m_{12}} p_1^{r+m_1+m_2+m_3+j}\left(\frac{1}{\beta}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{23}} p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_2} p_2^{r+m_1+i} \sum_{1\le j\le m_{13}} p_1^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\beta}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le r} p_3^k \sum_{1\le i\le m_2} p_2^{r+m_1+i} \sum_{1\le j\le r} p_1^j\left(\frac{1}{\beta}-\frac{1}{\delta^k}\right)+\sum_{1\le k\le r} p_3^k \sum_{1\le i\le m_{12}} p_2^{r+m_1+i} \sum_{1\le j\le r} p_1^j\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le m_{13}} p_3^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{23}} p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_{12}} p_1^{r+m_1+m_2+m_3+j}\left(\frac{1}{\delta^i}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{13}} p_3^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{23}} p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_{13}} p_1^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\delta^i}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{13}} p_3^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{23}} p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le r} p_1^j\left(\frac{1}{\delta^i}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{23}} p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_{23}} p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_{12}} p_1^{r+m_1+m_2+m_3+j}\left(\frac{1}{\delta^i}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{23}} p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_{23}} p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_{13}} p_1^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\delta^i}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{23}} p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_{23}} p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le r} p_1^j\left(\frac{1}{\delta^i}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le r} p_3^k \sum_{1\le i\le m_{23}} p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le r} p_1^j \left(\frac{1}{\delta^i}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le r} p_3^k \sum_{1\le i\le r} p_2^i \sum_{1\le j\le r} p_1^j \left(\frac{1}{\gamma^i}-\frac{1}{\gamma^k}\right)$$

Ο τρίτος όρος σύμφωνα με τον Πίνακα Β-3 αναλύεται ως εξής :

$$\sum_{1\le k<i<j\le m} p_3^k\, p_2^i\, p_1^j \left(\frac{1}{c^j}-\frac{1}{c^k}\right)=$$

$$\sum_{1\le k\le m_{13}} p_3^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{12}} p_2^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_1} p_1^{r+j} \left(\frac{1}{\gamma}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{13}} p_3^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le r} p_2^i \sum_{1\le j\le m_1} p_1^{r+j} \left(\frac{1}{\gamma}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{23}} p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_{12}} p_2^{r+m_1+m_2+m_3+m_{12}+i} \sum_{1\le j\le m_1} p_1^{r+j} \left(\frac{1}{\gamma}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{23}} p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_{23}} p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_1} p_1^{r+j} \left(\frac{1}{\gamma}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{23}} p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le r} p_2^i \sum_{1\le j\le m_1} p_1^{r+j} \left(\frac{1}{\gamma}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le r} p_3^k \sum_{1\le i\le r} p_2^i \sum_{1\le j\le m_1} p_1^{r+j} \left(\frac{1}{\gamma}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le r} p_3^k \sum_{1\le i\le r} p_2^i \sum_{1\le j\le m_{12}} p_1^{r+m_1+m_2+m_3+j} \left(\frac{1}{\zeta^j}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le m_{13}} p_3^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{12}} p_2^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_{13}} p_1^{r+m_1+m_2+m_3+m_{12}+j} \left(\frac{1}{\varepsilon^j}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{13}} p_3^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le r} p_2^i \sum_{1\le j\le m_{13}} p_1^{r+m_1+m_2+m_3+m_{12}+j} \left(\frac{1}{\varepsilon^j}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le r} p_3^k \sum_{1\le i\le r} p_2^i \sum_{1\le j\le m_{13}} p_1^{r+m_1+m_2+m_3+m_{12}+j} \left(\frac{1}{\varepsilon^j}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le r} p_3^k \sum_{1\le i\le r} p_2^i \sum_{1\le j\le r} p_1^j \left(\frac{1}{\gamma^j}-\frac{1}{\gamma^k}\right)$$

Ο τέταρτος όρος σύμφωνα με τον Πίνακα Β-4 αναλύεται ως εξής :

$$\sum_{1\le i<k<j\le m} p_3^k\, p_2^i\, p_1^j \left(\frac{1}{c^j}-\frac{1}{c^k}\right)=$$

$$\sum_{1\le k\le r} p_3^k \sum_{1\le i\le m_{12}} p_2^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_1} p_1^{r+j} \left(\frac{1}{\gamma}-\frac{1}{\gamma^{\kappa}}\right)$$

$$+\sum_{1\le k\le m_{13}} p_3^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{23}} p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_1} p_1^{r+j} \left(\frac{1}{\gamma}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{23}} p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_{23}} p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_1} p_1^{r+j} \left(\frac{1}{\gamma}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le r} p_3^k \sum_{1\le i\le m_{23}} p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_1} p_1^{r+j} \left(\frac{1}{\gamma}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le r} p_3^k \sum_{1\le i\le r} p_2^i \sum_{1\le j\le m_1} p_1^{r+j} \left(\frac{1}{\gamma}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le r} p_3^k \sum_{1\le i\le m_{12}} p_2^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_{12}} p_1^{r+m_1+m_2+m_3+j} \left(\frac{1}{\zeta^j}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le r} p_3^k \sum_{1\le i\le r} p_2^i \sum_{1\le j\le m_{12}} p_1^{r+m_1+m_2+m_3+j} \left(\frac{1}{\zeta^j}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le r} p_3^k \sum_{1\le i\le m_{12}} p_2^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_{13}} p_1^{r+m_1+m_2+m_3+m_{12}+j} \left(\frac{1}{\varepsilon^j}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le r} p_3^k \sum_{1\le i\le r} p_2^i \sum_{1\le j\le m_{13}} p_1^{r+m_1+m_2+m_3+m_{12}+j} \left(\frac{1}{\varepsilon^j}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le r} p_3^k \sum_{1\le i\le r} p_2^i \sum_{1\le j\le r} p_1^j \left(\frac{1}{\gamma^j}-\frac{1}{\gamma^k}\right)$$

Ο πέμπτος όρος σύμφωνα με τον Πίνακα Β-5 αναλύεται ως εξής :

$$\sum_{1\le i<k\le m} p_3^k\, p_2^i\, p_1^j \left(\frac{2}{c^i}-\frac{1}{c^k}\right)=$$

$$\sum_{1\le k\le m_3} p_3^{r+m_1+m_2+k} \sum_{1\le i\le m_{12}} p_2^{r+m_1+m_2+m_3+i} p_1^{r+m_1+m_2+m_3+i}\left(\frac{2}{\zeta^i}-\frac{1}{a}\right)$$

$$+\sum_{1\le k\le m_3} p_3^{r+m_1+m_2+k} \sum_{1\le i\le r} p_2^i p_1^i\left(\frac{2}{\gamma^i}-\frac{1}{a}\right)$$

$$+\sum_{1\le k\le m_{13}} p_3^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{12}} p_2^{r+m_1+m_2+m_3+i} p_1^{r+m_1+m_2+m_3+i}\left(\frac{2}{\zeta^i}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{13}} p_3^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le r} p_2^i p_1^i\left(\frac{2}{\gamma^i}-\frac{1}{\varepsilon^k}\right)$$

$$+\sum_{1\le k\le m_{23}} p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_{12}} p_2^{r+m_1+m_2+m_3+i} p_1^{r+m_1+m_2+m_3+i}\left(\frac{2}{\zeta^i}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le m_{23}} p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_{12}} p_2^i p_1^i\left(\frac{2}{\gamma^i}-\frac{1}{\delta^k}\right)$$

$$+\sum_{1\le k\le r} p_3^k \sum_{1\le i\le r} p_2^i p_1^i\left(\frac{2}{\gamma^i}-\frac{1}{\gamma^k}\right)$$

Ο έκτος όρος σύμφωνα με τον Πίνακα Β-6 αναλύεται ως εξής :

$$\sum_{1\le k<i\le m} p_3^k\, p_2^i\, p_1^j\left(\frac{2}{c^i}-\frac{1}{c^k}\right)=$$

$$\sum_{1\le k\le r} p_3^k \sum_{1\le i\le m_{12}} p_2^{r+m_1+m_2+m_3+i} p_1^{r+m_1+m_2+m_3+i}\left(\frac{2}{\zeta^i}-\frac{1}{\gamma^k}\right)$$

$$+\sum_{1\le k\le r} p_3^k \sum_{1\le i\le r} p_2^i p_1^i\left(\frac{2}{\zeta^i}-\frac{1}{\gamma^k}\right)$$

Το άθροισμα των 6 πιο πάνω όρων μας δίνει το Κοινωνικό Κόστος της Αυθαίρετης Ισορροπίας Nash σύμφωνα με το Λήμμα 4.1.

Κεφάλαιο 7

Υπολογισμός Πιθανοτήτων στην Αυθαίρετη Ισορροπία Nash και στην Πλήρως Μικτή Ισορροπία Nash

7.1 Πιθανότητες στην Αυθαίρετη Ισορροπία Nash

Εκμεταλλευόμαστε το γεγονός της Ισορροπίας Nash και χρησιμοποιούμε τις ανισότητες για να υπολογίσουμε τις πιθανότητες στην αυθαίρετη ισορροπία Nash. Αρχίζουμε με το χρήστη 1.

- Θεωρούμε οποιοδήποτε ζευγάρι από συνδέσεις $j, k \in support(2)$ με $j \in [r + m_1 + m_2 + m_3 + m_{12}] \backslash [r + m_1 + m_2 + m_3]$ και $k \in [r + m_1 + m_2] \backslash [r + m_1]$. Αφού P είναι ισορροπία Nash τότε ισχύει $IC(P)_2^j = IC(P)_2^k$. Επομένως $\frac{1+p_1^j}{\zeta^j} = \frac{1}{\beta}$, ή $p_1^j = \frac{\zeta^j}{\beta} - 1$.

- Θεωρούμε οποιοδήποτε ζευγάρι από συνδέσεις $j, k \in support(3)$ με $j \in [r + m_1 + m_2 + m_3 + m_{12} + m_{13}] \backslash [r + m_1 + m_2 + m_3 + m_{12}]$ και $k \in [r + m_1 + m_2 + m_3] \backslash [r + m_1 + m_2]$. Αφού P είναι ισορροπία Nash τότε ισχύει $IC(P)_3^j = IC(P)_3^k$. Επομένως $\frac{1+p_1^j}{\varepsilon^j} = \frac{1}{\alpha}$, ή $p_1^j = \frac{\varepsilon^j}{\alpha} - 1$.

- Θεωρούμε οποιοδήποτε ζευγάρι από συνδέσεις $j, k \in support(1)$ με $j \in [r]$ και $k \in [r + m_1] \backslash [r]$. Αφού P είναι ισορροπία Nash τότε ισχύει $IC(P)_1^j = IC(P)_1^k$. Επομένως $\frac{1+p_2^j+p_3^j}{\gamma^j} = \frac{1}{\gamma}$, ή $p_3^j = \frac{\gamma^j}{\gamma} - p_2^j - 1$. Οι επόμενες δυο σχέσεις θα μας βοηθήσουν να εκφράσουμε την p_1^j:

- Θεωρούμε οποιοδήποτε ζευγάρι από συνδέσεις $j,k \in support(3)$ με $j \in [r]$ και $k \in [r+m_1+m_2+m_3]\backslash[r+m_1+m_2]$. Αφού P είναι ισορροπία Nash τότε ισχύει $IC(P)_3^j = IC(P)_3^k$. Επομένως $\frac{1+p_1^j+p_2^j}{\gamma^j} = \frac{1}{\alpha}$, ή $p_2^j = \frac{\gamma^j}{\alpha} - p_1^j - 1$.

- Θεωρούμε οποιοδήποτε ζευγάρι από συνδέσεις $j,k \in support(2)$ με $j \in [r]$ και $k \in [r+m_1+m_2]\backslash[r+m_1]$. Αφού P είναι ισορροπία Nash τότε ισχύει $IC(P)_2^j = IC(P)_2^k$. Επομένως $\frac{1+p_1^j+p_3^j}{\gamma^j} = \frac{1}{\beta}$, ή $p_3^j = \frac{\gamma^j}{\beta} - p_1^j - 1$.

Επομένως $p_3^j = \frac{\gamma^j}{\gamma} - p_2^j - 1$ μπορεί να εκφραστεί ώς ακολούθως :

$$\frac{\gamma^j}{\beta} - p_1^j - 1 = \frac{\gamma^j}{\gamma} - p_2^j - 1$$

$$\frac{\gamma^j}{\beta} - p_1^j = \frac{\gamma^j}{\gamma} - \frac{\gamma^j}{\alpha} + p_1^j + 1$$

$$2p_1^j = \frac{\gamma^j}{\beta} - \frac{\gamma^j}{\gamma} + \frac{\gamma^j}{\alpha} - 1$$

$$p_1^j = \frac{\gamma^j}{2\beta} - \frac{\gamma^j}{2\gamma} + \frac{\gamma^j}{2\alpha} - \frac{1}{2}$$

- Εφόσον $support(1) = [r+m1]\backslash[r]$ και σύμφωνα με τα πιο πάνω συνεπάγεται ότι

$$\sum\nolimits_{j\in[r+m_1]\backslash[r]} p_1^j = 1 - \sum\nolimits_{j\in[r]} p_1^j - \sum\nolimits_{j\in[r+m_1+m_2+m_3+m_{12}]\backslash[r+m_1+m_2+m_3]} p_1^j - \sum\nolimits_{[r+m_1+m_2+m_3+m_{12}+m_{13}]\backslash[r+m_1+m_2+m_3+m_{12}]} p_1^j$$

$$= 1 - \sum\nolimits_{j\in[r]} \left(\frac{\gamma^j}{2\beta} - \frac{\gamma^j}{2\gamma} + \frac{\gamma^j}{2\alpha} - \frac{1}{2}\right) - \sum\nolimits_{j\in[r+m_1+m_2+m_3+m_{12}]\backslash[r+m_1+m_2+m_3]} \left(\frac{\zeta^j}{\beta} - 1\right) - \sum\nolimits_{[r+m_1+m_2+m_3+m_{12}+m_{13}]\backslash[r+m_1+m_2+m_3+m_{12}]} \left(\frac{\varepsilon^j}{\alpha} - 1\right)$$

$$= 1 - \frac{1}{2}\sum\nolimits_{j\in[r]}\left(\frac{\gamma^j}{\beta} - \frac{\gamma^j}{\gamma} + \frac{\gamma^j}{\alpha} - 1\right) - \sum\nolimits_{j\in[r+m_1+m_2+m_3+m_{12}]\backslash[r+m_1+m_2+m_3]}\left(\frac{\zeta^j - \beta}{\beta}\right) - \sum\nolimits_{[r+m_1+m_2+m_3+m_{12}+m_{13}]\backslash[r+m_1+m_2+m_3+m_{12}]}\left(\frac{\varepsilon^j - \alpha}{\alpha}\right)$$

$$= 1 - \frac{1}{2}\sum\nolimits_{j\in[r]}\left(\frac{\gamma^j - \beta}{\beta}\right) + \frac{1}{2}\sum\nolimits_{j\in[r]}\left(\frac{\gamma^j}{\gamma}\right) - \frac{1}{2}\sum\nolimits_{j\in[r]}\left(\frac{\gamma^j}{\alpha}\right) - \sum\nolimits_{j\in[r+m_1+m_2+m_3+m_{12}]\backslash[r+m_1+m_2+m_3]}\left(\frac{\zeta^j - \beta}{\beta}\right) - \sum\nolimits_{[r+m_1+m_2+m_3+m_{12}+m_{13}]\backslash[r+m_1+m_2+m_3+m_{12}]}\left(\frac{\varepsilon^j - \alpha}{\alpha}\right)$$

$$= 1 - \frac{\Gamma - r\beta}{2\beta} + \frac{\Gamma}{2\gamma} - \frac{\Gamma}{2\alpha} - \frac{Z - m_{12}\beta}{\beta} - \frac{E - m_{13}\alpha}{\alpha}$$

- Επομένως συνολικά οι πιθανότητες για το χρήστη 1στην P είναι:

$$p_1^j = \frac{\gamma^j}{2\beta} + \frac{\gamma^j}{2\alpha} - \frac{\gamma^j}{2\gamma} - \frac{1}{2} \text{ για όλες τις συνδέσεις } j \in [r]$$

$$p_1^{r+m_1+m_2+m_3+j} = \frac{\zeta^j}{\beta} - 1 \text{ για όλες τις συνδέσεις } j \in [m_{12}]$$

$$p_1^{r+m_1+m_2+m_3+m_{12}+j} = \frac{\varepsilon^j}{\alpha} - 1 \text{ για όλες τις συνδέσεις } j \in [m_{13}]$$

$$p_1^{r+m_1+m_2+m_3+m_{12}+m_{13}+j} = 0 \text{ για όλες τις συνδέσεις } j \in [m_{23}]$$

$$\sum\nolimits_{j\in[m_1]} p_1^{r+j} = \frac{(r+2)\beta - \Gamma}{2\beta} - \frac{Z - m_{12}\beta}{\beta} - \frac{E - m_{13}\alpha}{\alpha} + \frac{\Gamma(\alpha - \gamma)}{2\beta}$$

$$p_1^{r+m_1+j} = 0 \text{ για όλες τις συνδέσεις } j \in [m_2]$$

$$p_1^{r+m_1+m_2+j} = 0 \text{ για όλες τις συνδέσεις } j \in [m_3]$$

Συνεχίζουμε με το χρήστη 2:

- Θεωρούμε οποιοδήποτε ζευγάρι από συνδέσεις $j, k \in support(3)$ με $j \in [r + m_1 + m_2 + m_3 + m_{12} + m_{13}] \backslash [r + m_1 + m_2 + m_3 + m_{12}]$ και $k \in [r + m_1 + m_2 + m_3 + m_{12} + m_{13} + m_{23}] \backslash [r + m_1 + m_2 + m_3 + m_{12} + m_{13}]$. Αφού P είναι ισορροπία Nash τότε ισχύει $IC(P)_3^j = IC(P)_3^k$. Επομένως $\frac{1+p_1^j}{\varepsilon^j} = \frac{1+p_2^k}{\delta^j}$, ή $p_2^k = \frac{\delta^j}{\alpha} - 1$.

- Θεωρούμε οποιοδήποτε ζευγάρι από συνδέσεις $j, k \in support(1)$ με $j \in [r + m_1 + m_2 + m_3 + m_{12}] \backslash [r + m_1 + m_2 + m_3]$ και $k \in [r + m_1] \backslash [r]$. Αφού P είναι ισορροπία Nash τότε ισχύει $IC(P)_1^j = IC(P)_1^k$. Επομένως $\frac{1+p_2^j}{\zeta^j} = \frac{1}{\gamma}$, ή $p_2^j = \frac{\zeta^j}{\gamma} - 1$.

- Θεωρούμε οποιοδήποτε ζευγάρι από συνδέσεις $j, k \in support(3)$ με $j \in [r]$ και $k \in [r + m_1 + m_2 + m_3] \backslash [r + m_1 + m_2]$. Αφού P είναι ισορροπία Nash τότε ισχύει $IC(P)_3^j = IC(P)_3^k$. Επομένως $\frac{1+p_1^j+p_2^j}{\gamma^j} = \frac{1}{\alpha}$, ή $p_1^j = \frac{\gamma^j}{\alpha} - p_2^j - 1$. Οι επόμενες δυο σχέσεις θα μας βοηθήσουν να εκφράσουμε την p_2^j:

  - Θεωρούμε οποιοδήποτε ζευγάρι από συνδέσεις $j, k \in support(1)$ με $j \in [r]$ και $k \in [r + m_1] \backslash [r]$. Αφού P είναι ισορροπία Nash τότε ισχύει $IC(P)_1^j = IC(P)_1^k$. Επομένως $\frac{1+p_2^j+p_3^j}{\gamma^j} = \frac{1}{\gamma}$, ή $p_3^j = \frac{\gamma^j}{\gamma} - p_2^j - 1$.

  - Θεωρούμε οποιοδήποτε ζευγάρι από συνδέσεις $j, k \in support(2)$ με $j \in [r]$ και $k \in [r + m_1 + m_2] \backslash [r + m_1]$. Αφού P είναι ισορροπία Nash τότε ισχύει $IC(P)_2^j = IC(P)_2^k$. Επομένως $\frac{1+p_1^j+p_3^j}{\gamma^j} = \frac{1}{\beta}$, ή $p_1^j = \frac{\gamma^j}{\beta} - p_3^j - 1$.

  Επομένως $p_1^j = \frac{\gamma^j}{\alpha} - p_2^j - 1$ μπορεί να εκφραστεί ώς ακολούθως :

$$\frac{\gamma^j}{\beta} - p_3^j - 1 = \frac{\gamma^j}{\alpha} - p_2^j - 1$$

$$\frac{\gamma^j}{\beta} - \frac{\gamma^j}{\gamma} + p_2^j + 1 - 1 = \frac{\gamma^j}{\alpha} - p_2^j - 1$$

$$2p_2^j = \frac{\gamma^j}{\alpha} - \frac{\gamma^j}{\beta} + \frac{\gamma^j}{\gamma} - 1$$

$$p_2^j = \frac{\gamma^j}{2\alpha} - \frac{\gamma^j}{2\beta} + \frac{\gamma^j}{2\gamma} - \frac{1}{2}$$

- Εφόσον $support(2) = [r + m_1 + m_2]\backslash[r + m_1]$ και σύμφωνα με τα πιο πάνω συνεπάγεται ότι

$$\sum\nolimits_{j\in[r+m_1+m_2]\backslash[r+m_1]} p_2^j = 1 - \sum\nolimits_{j\in[r]} p_2^j - \sum\nolimits_{j\in[r+m_1+m_2+m_3+m_{12}]\backslash[r+m_1+m_2+m_3]} p_2^j - \sum\nolimits_{[r+m_1+m_2+m_3+m_{12}+m_{13}+m_{12}+m_{23}]\backslash[r+m_1+m_2+m_3+m_{12}+m_{13}]} p_2^j$$

$$= 1 - \sum\nolimits_{j\in[r]} \left(\frac{\gamma^j}{2\alpha} - \frac{\gamma^j}{2\beta} + \frac{\gamma^j}{2\gamma} - \frac{1}{2}\right) - \sum\nolimits_{j\in[r+m_1+m_2+m_3+m_{12}]\backslash[r+m_1+m_2+m_3]} \left(\frac{\zeta^j}{\gamma} - 1\right) - \sum\nolimits_{[r+m_1+m_2+m_3+m_{12}+m_{13}+m_{23}]\backslash[r+m_1+m_2+m_3+m_{12}+m_{13}]} \left(\frac{\delta^j}{\alpha} - 1\right)$$

$$= 1 - \frac{1}{2}\sum\nolimits_{j\in[r]} \left(\frac{\gamma^j}{\alpha} - \frac{\gamma^j}{\beta} + \frac{\gamma^j}{\gamma} - 1\right) - \sum\nolimits_{j\in[r+m_1+m_2+m_3+m_{12}]\backslash[r+m_1+m_2+m_3]} \left(\frac{\zeta^j - \gamma}{\gamma}\right) - \sum\nolimits_{[r+m_1+m_2+m_3+m_{12}+m_{13}+m_{23}]\backslash[r+m_1+m_2+m_3+m_{12}+m_{13}]} \left(\frac{\delta^j - \alpha}{\alpha}\right)$$

$$= 1 - \frac{1}{2}\sum\nolimits_{j\in[r]} \left(\frac{\gamma^j - \alpha}{\alpha}\right) + \frac{1}{2}\sum\nolimits_{j\in[r]} \left(\frac{\gamma^j}{\beta}\right) - \frac{1}{2}\sum\nolimits_{j\in[r]} \left(\frac{\gamma^j}{\gamma}\right) - \sum\nolimits_{j\in[r+m_1+m_2+m_3+m_{12}]\backslash[r+m_1+m_2+m_3]} \left(\frac{\zeta^j - \gamma}{\gamma}\right) - \sum\nolimits_{[r+m_1+m_2+m_3+m_{12}+m_{13}+m_{23}]\backslash[r+m_1+m_2+m_3+m_{12}+m_{13}]} \left(\frac{\delta^j - \alpha}{\alpha}\right)$$

$$= 1 - \frac{\Gamma - r\alpha}{2\alpha} + \frac{\Gamma}{2\beta} - \frac{\Gamma}{2\gamma} - \frac{Z - m_{12}\gamma}{\gamma} - \frac{\Delta - m_{23}\alpha}{\alpha}$$

- Επομένως συνολικά οι πιθανότητες για το χρήστη 2 στην P είναι:

$$p_2^j = \frac{\gamma^j}{2a} - \frac{\gamma^j}{2\beta} + \frac{\gamma^j}{2\gamma} - \frac{1}{2} \text{για όλες τις συνδέσεις } j \in [r]$$

$$p_2^{r+m_1+m_2+m_3+m_{12}+j} = \frac{\zeta^j}{\gamma} - 1 \text{ για όλες τις συνδέσεις } j \in [m_{12}]$$

$$p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+j} = \frac{\delta^j}{\alpha} - 1 \text{ για όλες τις συνδέσεις } j \in [m_{23}]$$

$$p_2^{r+m_1+m_2+m_3+m_{12}+j} = 0 \text{ για όλες τις συνδέσεις } j \in [m_{13}]$$

$$\sum_{j\in[m_2]} p_2^{r+m_1+j} = \frac{(r+2)\alpha - \Gamma}{2\alpha} - \frac{Z - m_{12}\gamma}{\gamma} - \frac{\Delta - m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma - \beta)}{2\beta}$$

$$p_2^{r+j} = 0 \text{ για όλες τις συνδέσεις } j \in [m_1]$$

$$p_2^{r+m_1+m_2+j} = 0 \text{ για όλες τις συνδέσεις } j \in [m_3]$$

Συνεχίζουμε με το χρήστη 3:

- Θεωρούμε οποιοδήποτε ζευγάρι από συνδέσεις $j, k \in support(2)$ με $j \in [r + m_1 + m_2 + m_3 + m_{12}] \backslash [r + m_1 + m_2 + m_3]$ και $k \in [r + m_1 + m_2 + m_3 + m_{12} + m_{13} + m_{23}] \backslash [r + m_1 + m_2 + m_3 + m_{12} + m_{13}]$. Αφού P είναι ισορροπία Nash τότε ισχύει $IC(P)_2^j = IC(P)_2^k$. Επομένως $\frac{1+p_1^j}{\zeta^j} = \frac{1+p_3^k}{\delta^j}$, ή $p_3^k = \frac{\delta^j}{\beta} - 1$.

- Θεωρούμε οποιοδήποτε ζευγάρι από συνδέσεις $j, k \in support(1)$ με $j \in [r + m_1 + m_2 + m_3 + m_{12} + m_{13}] \backslash [r + m_1 + m_2 + m_3 + m_{12}]$ και $k \in [r + m_1] \backslash [r]$. Αφού P είναι ισορροπία Nash τότε ισχύει $IC(P)_1^j = IC(P)_1^k$. Επομένως $\frac{1+p_3^j}{\varepsilon^j} = \frac{1}{\gamma}$, ή $p_3^j = \frac{\varepsilon^j}{\gamma} - 1$.

- Θεωρούμε οποιοδήποτε ζευγάρι από συνδέσεις $j, k \in support(3)$ με $j \in [r]$ και $k \in [r + m_1 + m_2 + m_3] \backslash [r + m_1 + m_2]$. Αφού P είναι ισορροπία Nash τότε ισχύει $IC(P)_3^j = IC(P)_3^k$. Επομένως $\frac{1+p_1^j+p_2^j}{\gamma^j} = \frac{1}{\alpha}$, ή $p_1^j = \frac{\gamma^j}{\alpha} - p_2^j - 1$. Οι επόμενες δυο σχέσεις θα μας βοηθήσουν να εκφράσουμε την p_3^j:

    - Θεωρούμε οποιοδήποτε ζευγάρι από συνδέσεις $j, k \in support(1)$ με $j \in [r]$ και $k \in [r + m_1] \backslash [r]$. Αφού P είναι ισορροπία Nash τότε ισχύει $IC(P)_1^j = IC(P)_1^k$. Επομένως $\frac{1+p_2^j+p_3^j}{\gamma^j} = \frac{1}{\gamma}$, ή $p_2^j = \frac{\gamma^j}{\gamma} - p_3^j - 1$.

    - Θεωρούμε οποιοδήποτε ζευγάρι από συνδέσεις $j, k \in support(2)$ με $j \in [r]$ και $k \in [r + m_1 + m_2] \backslash [r + m_1]$. Αφού P είναι ισορροπία Nash τότε ισχύει $IC(P)_2^j = IC(P)_2^k$. Επομένως $\frac{1+p_1^j+p_3^j}{\gamma^j} = \frac{1}{\beta}$, ή $p_1^j = \frac{\gamma^j}{\beta} - p_3^j - 1$.

Επομένως $p_1^j = \frac{\gamma^j}{\alpha} - p_2^j - 1$ μπορεί να εκφραστεί ώς ακολούθως :

$$\frac{\gamma^j}{\beta} - p_3^j - 1 = \frac{\gamma^j}{\alpha} - p_2^j - 1$$

$$\frac{\gamma^j}{\beta} - p_3^j - 1 = \frac{\gamma^j}{\alpha} - \frac{\gamma^j}{\gamma} + p_3^j + 1 - 1$$

$$2p_3^j = \frac{\gamma^j}{\beta} - \frac{\gamma^j}{\alpha} + \frac{\gamma^j}{\gamma} - 1$$

$$p_3^j = \frac{\gamma^j}{2\beta} - \frac{\gamma^j}{2\alpha} + \frac{\gamma^j}{2\gamma} - \frac{1}{2}$$

- Εφόσον $support(3) = [r + m_1 + m_2 + m_3]\backslash[r + m_1 + m_2]$ και σύμφωνα με τα πιο πάνω συνεπάγεται ότι

$$\sum\nolimits_{j\in[r+m_1+m_2]\backslash[r+m_1]} p_3^j = 1 - \sum\nolimits_{j\in[r]} p_3^j - \sum\nolimits_{j\in[r+m_1+m_2+m_3+m_{12}+m_{13}]\backslash[r+m_1+m_2+m_3+m_{12}]} p_3^j - \sum\nolimits_{[r+m_1+m_2+m_3+m_{12}+m_{13}+m_{23}]\backslash[r+m_1+m_2+m_3+m_{12}+m_{13}]} p_3^j$$

$$= 1 - \sum\nolimits_{j\in[r]} \left(\frac{\gamma^j}{2\beta} - \frac{\gamma^j}{2\alpha} + \frac{\gamma^j}{2\gamma} - \frac{1}{2}\right) - \sum\nolimits_{j\in[r+m_1+m_2+m_3+m_{12}]\backslash[r+m_1+m_2+m_3]} \left(\frac{\varepsilon^j}{\gamma} - 1\right) - \sum\nolimits_{[r+m_1+m_2+m_3+m_{12}+m_{13}+m_{23}]\backslash[r+m_1+m_2+m_3+m_{12}+m_{13}]} \left(\frac{\delta^j}{\beta} - 1\right)$$

$$= 1 - \frac{1}{2}\sum\nolimits_{j\in[r]} \left(\frac{\gamma^j}{\beta} - \frac{\gamma^j}{\alpha} + \frac{\gamma^j}{\gamma} - 1\right) - \sum\nolimits_{j\in[r+m_1+m_2+m_3+m_{12}]\backslash[r+m_1+m_2+m_3]} \left(\frac{\varepsilon^j - \gamma}{\gamma}\right) - \sum\nolimits_{[r+m_1+m_2+m_3+m_{12}+m_{13}+m_{12}+m_{23}]\backslash[r+m_1+m_2+m_3+m_{12}+m_{13}]} \left(\frac{\delta^j - \beta}{\beta}\right)$$

$$= 1 - \frac{1}{2}\sum\nolimits_{j\in[r]} \left(\frac{\gamma^j - \beta}{\beta}\right) + \frac{1}{2}\sum\nolimits_{j\in[r]} \left(\frac{\gamma^j}{\alpha}\right) - \frac{1}{2}\sum\nolimits_{j\in[r]} \left(\frac{\gamma^j}{\gamma}\right) - \sum\nolimits_{j\in[r+m_1+m_2+m_3+m_{12}]\backslash[r+m_1+m_2+m_3]} \left(\frac{\varepsilon^j - \gamma}{\gamma}\right) - \sum\nolimits_{[r+m_1+m_2+m_3+m_{12}+m_{13}+m_{23}]\backslash[r+m_1+m_2+m_3+m_{12}+m_{13}]} \left(\frac{\delta^j - \beta}{\beta}\right)$$

$$= 1 - \frac{\Gamma - r\beta}{2\beta} + \frac{\Gamma}{2\alpha} - \frac{\Gamma}{2\gamma} - \frac{E - m_{13}\gamma}{\gamma} - \frac{\Delta - m_{23}\beta}{\beta}$$

- Επομένως συνολικά οι πιθανότητες για το χρήστη 3 στην P είναι:

$$p_3^j = \frac{\gamma^j}{2\beta} - \frac{\gamma^j}{2\alpha} + \frac{\gamma^j}{2\gamma} - \frac{1}{2} \text{για όλες τις συνδέσεις } j \in [r]$$

$$p_3^{r+m_1+m_2+m_3+j} = 0 \ \text{για όλες τις συνδέσεις } j \in [m_{12}]$$

$$p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+j} = \frac{\delta^j}{\beta} - 1 \ \text{για όλες τις συνδέσεις } j \in [m_{23}]$$

$$p_3^{r+m_1+m_2+m_3+m_{12}+j} = \frac{\varepsilon^j}{\gamma} - 1 \ \text{για όλες τις συνδέσεις } j \in [m_{13}]$$

$$\sum_{j \in [m_3]} p_3^{r+m_1+j} = \frac{(r+2)\beta - \Gamma}{2\beta} - \frac{E - m_{13}\gamma}{\gamma} - \frac{\Delta - m_{23}\beta}{\beta} + \frac{\Gamma(\gamma - \alpha)}{2\alpha\gamma}$$

$$p_3^{r+j} = 0 \ \text{για όλες τις συνδέσεις } j \in [m_1]$$

$$p_3^{r+m_1+m_2+j} = 0 \ \text{για όλες τις συνδέσεις } j \in [m_3]$$

7.2 Πιθανότητες στην Πλήρως Μικτή Ισορροπία Nash

Πιο κάτω υπολογίζουμε τις πιθανότητες στην Πλήρως Μικτή Ισορροπία Nash οι οποίες υπολογίζονται συναρτήση των πιθανοτήτων στην Αυθαίρετη Ισορροπία Nash.

Απαραίτητες και επαρκείς συνθήκες για την ύπαρξη της P είναι πως $0 \leq 1 - \frac{\Gamma - r\beta}{2\beta} - \frac{\Gamma}{2\alpha} + \frac{\Gamma}{2\gamma} - \frac{Z - m_{12}\beta}{\beta} - \frac{E - m_{13}\alpha}{\alpha} \leq 1$, $0 \leq 1 - \frac{\Gamma - r\alpha}{2\alpha} + \frac{\Gamma}{2\beta} - \frac{\Gamma}{2\gamma} - \frac{Z - m_{12}\gamma}{\gamma} - \frac{\Delta - m_{13}\alpha}{\alpha} \leq 1$ και $0 \leq 1 - \frac{\Gamma - r\beta}{2\beta} + \frac{\Gamma}{2\alpha} - \frac{\Gamma}{2\gamma} - \frac{E - m_{13}\gamma}{\gamma} - \frac{\Delta - m_{23}\beta}{\beta} \leq 1$.

Για κάθε σύνδεση $i \in [r]$,

$$f^i - p_1^i = \frac{(m+2)\gamma^i}{2C} - \frac{1}{2} - \frac{\gamma^i}{2\beta} - \frac{\gamma^i}{2\alpha} + \frac{\gamma^i}{2\gamma} + \frac{1}{2} =$$

$$\frac{\gamma^i}{2}\left(\frac{(m+2)}{C} - \frac{1}{\beta} - \frac{1}{\alpha} + \frac{1}{\gamma}\right) = \frac{\gamma^i}{2}\left(B - \frac{1}{\alpha} + \frac{1}{\gamma}\right) = \frac{1}{2}\left(\gamma^i B + \frac{\gamma^i}{\gamma} - \frac{\gamma^i}{\alpha}\right)$$

$$\rightarrow f^i = \tfrac{1}{2}\left(\gamma^i B + \tfrac{\gamma^i}{\beta} - 1\right),$$

$$f^i - p_2^i = \frac{(m+2)\gamma^i}{2C} - \frac{1}{2} - \frac{\gamma^i}{2a} + \frac{\gamma^i}{2\beta} - \frac{\gamma^i}{2\gamma} + \frac{1}{2} =$$

$$\frac{\gamma^i}{2}\left(\frac{(m+2)}{C} - \frac{1}{\alpha} + \frac{1}{\beta} - \frac{1}{\gamma}\right) = \frac{\gamma^i}{2}\left(A + \frac{1}{\beta} - \frac{1}{\gamma}\right) = \frac{1}{2}\left(\gamma^i A + \frac{\gamma^i}{\beta} - \frac{\gamma^i}{\gamma}\right)$$

$$\rightarrow f^i = \frac{1}{2}\left(\gamma^i A + \frac{\gamma^i}{\beta} - \frac{\gamma^i}{\gamma}\right) + \frac{\gamma^i}{2a} - \frac{\gamma^i}{2\beta} + \frac{\gamma^i}{2\gamma} - \frac{1}{2} = \frac{\gamma^i A}{2} + \frac{\gamma^i}{2a} - \frac{1}{2} = \frac{1}{2}\left(\gamma^i A + \frac{\gamma^i}{a} - 1\right)$$

και,

$$f^i - p_3^i = \frac{(m+2)\gamma^i}{2C} - \frac{1}{2} - \frac{\gamma^i}{2\beta} + \frac{\gamma^i}{2\alpha} - \frac{\gamma^i}{2\gamma} + \frac{1}{2} =$$

$$\frac{\gamma^i}{2}\left(\frac{(m+2)}{C} - \frac{1}{\beta} + \frac{1}{\alpha} - \frac{1}{\gamma}\right) = \frac{\gamma^i}{2}\left(B + \frac{1}{\alpha} - \frac{1}{\gamma}\right) = \frac{1}{2}\left(\gamma^i B + \frac{\gamma^i}{\alpha} - \frac{\gamma^i}{\gamma}\right)$$

$$\rightarrow f^i = \frac{1}{2}\left(\gamma^i B + \frac{\gamma^i}{\alpha} - \frac{\gamma^i}{\gamma}\right) + p_3^i = \frac{1}{2}\left(\gamma^k B + \frac{\gamma^i}{\alpha} - \frac{\gamma^i}{\gamma}\right) + \frac{\gamma^i}{2\beta} - \frac{\gamma^i}{2\alpha} + \frac{\gamma^i}{2\gamma} - \frac{1}{2}$$
$$= \frac{\gamma^i B}{2} + \frac{\gamma^i}{2\beta} - \frac{1}{2} = \frac{1}{2}\left(\gamma^i B + \frac{\gamma^i}{\beta} - 1\right)$$

για κάθε σύνδεση $i \in [r + m_1 + m_2 + m_3 + m_{12}] \backslash [r + m_1 + m_2 + m_3]$,

$$f^i - p_1^i = \frac{(m+2)\zeta^i}{2C} - \frac{1}{2} - \frac{\zeta^i}{\beta} + 1 = \frac{(m+2)\zeta^i}{2C} - \frac{\zeta^i}{\beta} + \frac{1}{2} =$$

$$\zeta^i\left(\frac{(m+2)}{2C} - \frac{1}{\beta}\right) + \frac{1}{2} = \zeta^i\left(\frac{(m+2)\beta - 2C}{2\beta C}\right) + \frac{1}{2} =$$

$$\zeta^i\left(\frac{1}{2}\left(\frac{(m+2)\beta - C}{\beta C} - \frac{1}{\beta}\right)\right) + \frac{1}{2} = \zeta^i\left(\frac{B}{2} - \frac{1}{2\beta}\right) + \frac{1}{2} = \frac{1}{2}\left(B\zeta^i - p_1^i\right)$$

$$\rightarrow f^i = \frac{1}{2}\left(B\zeta^i - p_1^i\right) + p_1^i = \frac{1}{2}\left(B\zeta^i + p_1^i\right) = \frac{1}{2}\left(B\zeta^i + p_1^i\right)$$

και,

$$f^i - p_2^i = \frac{(m+2)\zeta^i}{2C} - \frac{1}{2} - \frac{\zeta^i}{\gamma} + 1 = \frac{(m+2)\zeta^i}{2C} - \frac{\zeta^i}{\gamma} + \frac{1}{2} =$$

$$\zeta^i\left(\frac{(m+2)}{2C} - \frac{1}{\gamma}\right) + \frac{1}{2} = \zeta^i\left(\frac{(m+2)\gamma - 2C}{2\gamma C}\right) + \frac{1}{2} =$$

$$\zeta^i\left(\frac{1}{2}\left(\frac{(m+2)\gamma - C}{\gamma C} - \frac{1}{\gamma}\right)\right) + \frac{1}{2} = \zeta^i\left(\frac{G}{2} - \frac{1}{2\gamma}\right) + \frac{1}{2} = \frac{1}{2}\left(G\zeta^i - p_2^i\right)$$

$$\rightarrow f^i = \frac{1}{2}\left(G\zeta^i + p_2^i\right)$$

για κάθε σύνδεση $i \in [r + m_1 + m_2 + m_3 + m_{12} + m_{13}] \backslash [r + m_1 + m_2 + m_3 + m_{12}]$,

$$f^i - p_1^i = \frac{(m+2)\varepsilon^i}{2C} - \frac{1}{2} - \frac{\varepsilon^i}{\alpha} + 1 = \frac{(m+2)\varepsilon^j}{2C} - \frac{\varepsilon^j}{\alpha} + \frac{1}{2} =$$

$$\varepsilon^j\left(\frac{(m+2)}{2C} - \frac{1}{\alpha}\right) + \frac{1}{2} = \varepsilon^j\left(\frac{(m+2)\alpha - 2C}{2\alpha C}\right) + \frac{1}{2} =$$

$$\varepsilon^j\left(\frac{1}{2}\left(\frac{(m+2)\alpha - C}{\alpha C} - \frac{1}{\alpha}\right)\right) + \frac{1}{2} = \varepsilon^j\left(\frac{A}{2} - \frac{1}{2\alpha}\right) + \frac{1}{2} = \frac{1}{2}\left(A\varepsilon^j - p_1^i\right)$$

$$\rightarrow f^i = \frac{1}{2}\left(A\varepsilon^j - \frac{\varepsilon^j}{\alpha} + 1\right) + \frac{\varepsilon^j}{\alpha} - 1 = \frac{1}{2}\left(A\varepsilon^j + p_1^i\right)$$

και,

$$f^i - p_3^i = \frac{(m+2)\varepsilon^i}{2C} - \frac{1}{2} - \frac{\varepsilon^i}{\gamma} + 1 = \frac{(m+2)\varepsilon^i}{2C} - \frac{\varepsilon^i}{\gamma} + \frac{1}{2} =$$

$$\varepsilon^j\left(\frac{(m+2)}{2C} - \frac{1}{\gamma}\right) + \frac{1}{2} = \varepsilon^i\left(\frac{(m+2)\gamma - 2C}{2\gamma C}\right) + \frac{1}{2} =$$

$$\varepsilon^i\left(\frac{1}{2}\left(\frac{(m+2)\gamma - C}{\gamma C} - \frac{1}{\gamma}\right)\right) + \frac{1}{2} = \varepsilon^i\left(\frac{G}{2} - \frac{1}{2\gamma}\right) + \frac{1}{2} = \frac{1}{2}\left(G\varepsilon^i - p_3^i\right),$$

$$\rightarrow f^i = \frac{1}{2}\left(G\varepsilon^i + p_3^i\right)$$

για κάθε σύνδεση $i \in [r + m_1 + m_2 + m_3 + m_{12} + m_{13} + m_{23}] \backslash [r + m_1 + m_2 + m_3 + m_{12} + m_{13}]$,

$$f^i - p_2^i = \frac{(m+2)\delta^i}{2C} - \frac{1}{2} - \frac{\delta^i}{\alpha} + 1 = \frac{(m+2)\delta^i}{2C} - \frac{\delta^i}{\alpha} + \frac{1}{2} =$$

$$\delta^j\left(\frac{(m+2)}{2C}-\frac{1}{\alpha}\right)+\frac{1}{2}=\delta^i\left(\frac{(m+2)\alpha-2C}{2\alpha C}\right)+\frac{1}{2}=$$

$$\delta^j\left(\frac{1}{2}\left(\frac{(m+2)\alpha-C}{\alpha C}-\frac{1}{\alpha}\right)\right)+\frac{1}{2}=\delta^i\left(\frac{A}{2}-\frac{1}{2\alpha}\right)+\frac{1}{2}=\frac{1}{2}\left(A\delta^i-p_2^i\right)$$

$$\rightarrow f^i=\frac{1}{2}\left(A\delta^i+p_2^i\right)$$

και,

$$f^i-p_3^i=\frac{(m+2)\delta^i}{2C}-\frac{1}{2}-\frac{\delta^i}{\beta}+1\ =\frac{(m+2)\delta^i}{2C}-\frac{\delta^i}{\beta}+\frac{1}{2}=$$

$$\delta^i\left(\frac{(m+2)}{2C}-\frac{1}{\beta}\right)+\frac{1}{2}=\delta^i\left(\frac{(m+2)\beta-2C}{2\beta C}\right)+\frac{1}{2}=$$

$$\delta^i\left(\frac{1}{2}\left(\frac{(m+2)\beta-C}{\beta C}-\frac{1}{\gamma}\right)\right)+\frac{1}{2}=\delta^i\left(\frac{B}{2}-\frac{1}{2\beta}\right)+\frac{1}{2}=\frac{1}{2}\left(B\delta^i-p_3^i\right)$$

$$\rightarrow f^i=\frac{1}{2}\left(B\delta^i+p_3^i\right).$$

Επίσης έχουμε ότι

$f^{r+i}=\frac{(m+2)\gamma-C}{C}=\gamma\,\frac{(m+2)\gamma-C}{\gamma C}=\gamma G,$ για κάθε σύνδεση $i\in[m_1]$,

$f^{r+m_1+i}=\frac{(m+2)\beta-C}{C}=\beta\,\frac{(m+2)\beta-C}{\beta C}=\beta B,$ για κάθε σύνδεση $i\in[m_2]$,

$f^{r+m_1+m_2+i}=\frac{(m+2)\gamma-C}{C}=\alpha\,\frac{(m+2)\alpha-C}{\alpha C}=\alpha A,$ για κάθε σύνδεση $i\in[m_3]$

Κεφάλαιο 8

Η Διαφορά Δ της Πλήρως Μικτής Ισορροπίας Nash και της Αυθαίρετης Ισορροπίας Nash

8.1 Η Διαφορά Δ

Στα Κεφάλαια 5 και 6 είδαμε τις δυο εκφράσεις του Κοινωνικού Κόστους όσον αφορά την περίπτωση της Πλήρως Μικτής Ισορροπίας Nash $SC(F)$ και την περίπτωση της Αυθαίρετης Ισορροπίας Nash $SC(P)$. Θα προσπαθήσουμε να αποδείξουμε το **θεώρημα 3.1** το οποίο λέει $\Delta = SC(F) - SC(P) \geq 0$. Μια πολύ σημαντική παρατήρηση, είναι ότι οι περιπτώσεις που έχουμε στην περίπτωση της Αυθαίρετης Ισορροπίας Nash αποτελούν **υποσύνολο** των περιπτώσεων της Πλήρως Μικτής Ισορροπίας Nash. Με τους πίνακες στα Παραρτήματα Α και Β μπορούμε να συνδυάσουμε εύκολα τις περιπτώσεις αυτές, παίρνοντας τις περιπτώσεις όπου οι δείκτες (i, k και j) είναι ίδιοι. Έτσι δημιουργούνται οι διαφορές Δ_2, Δ_3 ...,Δ_{60} που βλέπουμε πιο κάτω, ενώ οι $\Delta_{61}, \Delta_{62}, \dots, \Delta_{303}$ είναι αυτές που υπολείπονται από την Πλήρως Μικτή Ισορροπία Nash.

Έτσι έχουμε ότι $SC(F) - SC(P) = \Delta_1 + \Delta_2 + \cdots + \Delta_{303}$, όπου θα χρειαστεί να αποδείξουμε ότι $\Delta_1 + \Delta_2 + \cdots + \Delta_{303} \geq 0$ έτσι ώστε $\Delta \geq 0$, για την απόδειξη του **θεωρήματος 3.1**. Πιο κάτω ακολουθούν οι διαφορές $\Delta_1, \Delta_2, \dots, \Delta_{60}$, αφού οι υπόλοιπες της ΠΜΙΝ παραμένουν αυτές που είδαμε και πιο πριν:

$$\Delta_1 = IC_3^1(F) - IC_3^1(P) = \frac{1+2f^1}{\gamma^1} - \frac{1+p_1^1+p_2^1}{\gamma^1}$$

$$\Delta_2 = \sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le m_1} f^{r+j}\left(\frac{1}{\beta}-\frac{1}{\delta^k}\right)$$
$$- \sum_{1\le k\le m_{23}} p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_2} p_2^{r+m_1+i} \sum_{1\le j\le m_1} p_1^{r+j}\left(\frac{1}{\beta}-\frac{1}{\delta^k}\right)$$

$$\Delta_3 = \sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le m_1} f^{r+j}\left(\frac{1}{\beta}-\frac{1}{\varepsilon^k}\right)$$
$$- \sum_{1\le k\le m_{13}} p_3^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_2} p_2^{r+m_1+i} \sum_{1\le j\le m_1} p_1^{r+j}\left(\frac{1}{\beta}-\frac{1}{\varepsilon^k}\right)$$

$$\Delta_4 = \sum_{1\le k\le r} f^k \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le m_1} f^{r+j}\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right) - \sum_{1\le k\le r} p_3^k \sum_{1\le i\le m_2} p_2^{r+m_1+i} \sum_{1\le j\le m_1} p_1^{r+j}\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)$$

$$\Delta_5 = \sum_{1\le k\le r} f^k \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j}\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)$$
$$- \sum_{1\le k\le r} p_3^k \sum_{1\le i\le m_2} p_2^{r+m_1+i} \sum_{1\le j\le m_{12}} p_1^{r+m_1+m_2+m_3+j}\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)$$

$$\Delta_6 = \sum_{1\le k\le r} f^k \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)$$
$$- \sum_{1\le k\le r} p_3^k \sum_{1\le i\le m_2} p_2^{r+m_1+i} \sum_{1\le j\le m_{13}} p_1^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)$$

$$\Delta_7 = \sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\beta}-\frac{1}{\varepsilon^k}\right)$$
$$- \sum_{1\le k\le m_{13}} p_3^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_2} p_2^{r+m_1+i} \sum_{1\le j\le m_{13}} p_1^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\beta}-\frac{1}{\varepsilon^k}\right)$$

$$\Delta_8 = \sum_{1\le k\le r} f^k \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le r} f^j \left(\frac{1}{\beta} - \frac{1}{\gamma^k}\right) - \sum_{1\le k\le r} p_3^k \sum_{1\le i\le m_2} p_2^{r+m_1+i} \sum_{1\le j\le r} p_1^j \left(\frac{1}{\beta} - \frac{1}{\gamma^k}\right)$$

$$\Delta_9 = \sum_{1\le k\le r} f^k \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j} \left(\frac{1}{\zeta^i} - \frac{1}{\gamma^k}\right)$$
$$- \sum_{1\le k\le r} p_3^k \sum_{1\le i\le m_{12}} p_2^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_{12}} p_1^{r+m_1+m_2+m_3+j} \left(\frac{1}{\zeta^i} - \frac{1}{\gamma^k}\right)$$

$$\Delta_{10} = \sum_{1\le k\le r} f^k \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\le j\le r} f^j \left(\frac{1}{\zeta^i} - \frac{1}{\gamma^k}\right)$$
$$- \sum_{1\le k\le r} p_3^k \sum_{1\le i\le m_{12}} p_2^{r+m_1+m_2+m_3+i} \sum_{1\le j\le r} p_1^j \left(\frac{1}{\zeta^i} - \frac{1}{\gamma^k}\right)$$

$$\Delta_{11} = \sum_{1\le k\le r} f^k \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j} \left(\frac{1}{\delta^i} - \frac{1}{\gamma^k}\right)$$
$$- \sum_{1\le k\le r} p_3^k \sum_{1\le i\le m_{23}} p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_{12}} p_1^{r+m_1+m_2+m_3+j} \left(\frac{1}{\delta^i} - \frac{1}{\gamma^k}\right)$$

$$\Delta_{12} = \sum_{1\le k\le r} f^k \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j} \left(\frac{1}{\delta^i} - \frac{1}{\gamma^k}\right)$$
$$- \sum_{1\le k\le r} p_3^k \sum_{1\le i\le m_{23}} p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_{13}} p_1^{r+m_1+m_2+m_3+m_{12}+j} \left(\frac{1}{\delta^i} - \frac{1}{\gamma^k}\right)$$

$$\Delta_{13} = \sum_{1\le k\le r} f^k \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le r} f^j \left(\frac{1}{\delta^i} - \frac{1}{\gamma^k}\right)$$
$$- \sum_{1\le k\le r} p_3^k \sum_{1\le i\le m_{23}} p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le r} p_1^j \left(\frac{1}{\delta^i} - \frac{1}{\gamma^k}\right)$$

$$\Delta_{14} = \sum_{1\le k\le r} f^k \sum_{1\le i\le r} f^i \sum_{1\le j\le r} f^j \left(\frac{1}{\gamma^i} - \frac{1}{\gamma^k}\right) - \sum_{1\le k\le r} p_3^k \sum_{1\le i\le r} p_2^i \sum_{1\le j\le r} p_1^j \left(\frac{1}{\gamma^i} - \frac{1}{\gamma^k}\right)$$

$$\Delta_{15} = \sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j} \left(\frac{1}{\beta} - \frac{1}{\varepsilon^k}\right)$$
$$- \sum_{1\le k\le m_{13}} p_3^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_2} p_2^{r+m_1+i} \sum_{1\le j\le m_{12}} p_1^{r+m_1+m_2+m_3+j} \left(\frac{1}{\beta} - \frac{1}{\varepsilon^k}\right)$$

$$\Delta_{16} = \sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j} \left(\frac{1}{\beta} - \frac{1}{\varepsilon^k}\right)$$
$$- \sum_{1\le k\le m_{13}} p_3^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_2} p_2^{r+m_1+i} \sum_{1\le j\le m_{13}} p_1^{r+m_1+m_2+m_3+m_{12}+j} \left(\frac{1}{\beta} - \frac{1}{\varepsilon^k}\right)$$

$$\Delta_{17} = \sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le r} f^j \left(\frac{1}{\beta} - \frac{1}{\varepsilon^k}\right)$$
$$- \sum_{1\le k\le m_{13}} p_3^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_2} p_2^{r+m_1+i} \sum_{1\le j\le r} p_1^j \left(\frac{1}{\beta} - \frac{1}{\varepsilon^k}\right)$$

$$\Delta_{18} = \sum_{1\le k\le m_{23}} f^{+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le r} f^j \left(\frac{1}{\beta} - \frac{1}{\delta^k}\right)$$
$$- \sum_{1\le k\le m_{23}} p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_2} p_2^{r+m_1+i} \sum_{1\le j\le r} p_1^j \left(\frac{1}{\beta} - \frac{1}{\delta^k}\right)$$

$$\Delta_{19} = \sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j} \left(\frac{1}{\beta} - \frac{1}{\delta^k}\right)$$
$$- \sum_{1\le k\le m_{23}} p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_2} p_2^{r+m_1+i} \sum_{1\le j\le m_{12}} p_1^{r+m_1+m_2+m_3+j} \left(\frac{1}{\beta} - \frac{1}{\delta^k}\right)$$

$$\Delta_{20} = \sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\beta}-\frac{1}{\delta^k}\right)$$

$$- \sum_{1\le k\le m_{23}} p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_2} p_2^{r+m_1+i} \sum_{1\le j\le m_{13}} p_1^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\beta}-\frac{1}{\delta^k}\right)$$

$$\Delta_{21} = \sum_{1\le k\le r} f^k \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le r} f^j\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right) - \sum_{1\le k\le r} p_3^k \sum_{1\le i\le m_2} p_2^{r+m_1+i} \sum_{1\le j\le r} p_1^j\left(\frac{1}{\beta}-\frac{1}{\delta^k}\right)$$

$$\Delta_{22} = \sum_{1\le k\le r} f^k \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\le j\le r} f^j\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^k}\right)$$

$$- \sum_{1\le k\le r} p_3^k \sum_{1\le i\le m_{12}} p_2^{r+m_1+m_2+m_3+i} \sum_{1\le j\le r} p_1^j\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^k}\right)$$

$$\Delta_{23} = \sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j}\left(\frac{1}{\delta^i}-\frac{1}{\varepsilon^k}\right)$$

$$- \sum_{1\le k\le m_{13}} p_3^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{23}} p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_{12}} p_1^{r+m_1+m_2+m_3+j}\left(\frac{1}{\delta^i}-\frac{1}{\varepsilon^k}\right)$$

$$\Delta_{24} = \sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\delta^i}-\frac{1}{\varepsilon^k}\right)$$

$$- \sum_{1\le k\le m_{13}} p_3^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{23}} p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_{13}} p_1^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\delta^i}-\frac{1}{\varepsilon^k}\right)$$

$$\Delta_{25} = \sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le r} f^j\left(\frac{1}{\delta^i}-\frac{1}{\varepsilon^k}\right)$$

$$- \sum_{1\le k\le m_{13}} p_3^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{23}} p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le r} p_1^j\left(\frac{1}{\delta^i}-\frac{1}{\varepsilon^k}\right)$$

$$\Delta_{26} = \sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j}\left(\frac{1}{\delta^i}-\frac{1}{\delta^k}\right)$$

$$-\sum_{1\le k\le m_{23}} p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_{23}} p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_{12}} p_1^{r+m_1+m_2+m_3+j}\left(\frac{1}{\delta^i}-\frac{1}{\delta^k}\right)$$

$$\Delta_{27} = \sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\delta^i}-\frac{1}{\delta^k}\right)$$

$$-\sum_{1\le k\le m_{23}} p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_{23}} p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_{13}} p_1^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\delta^i}-\frac{1}{\delta^k}\right)$$

$$\Delta_{28} = \sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le r} f^{j}\left(\frac{1}{\delta^i}-\frac{1}{\delta^k}\right)$$

$$-\sum_{1\le k\le m_{23}} p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_{23}} p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le r} p_1^{j}\left(\frac{1}{\delta^i}-\frac{1}{\delta^k}\right)$$

$$\Delta_{29} = \sum_{1\le k\le r} f^{k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le r} f^{j}\left(\frac{1}{\delta^i}-\frac{1}{\gamma^k}\right)$$

$$-\sum_{1\le k\le r} p_3^{k} \sum_{1\le i\le m_{23}} p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le r} p_1^{j}\left(\frac{1}{\delta^i}-\frac{1}{\gamma^k}\right)$$

$$\Delta_{30} = \sum_{1\le k\le r} f^{k} \sum_{1\le i\le r} f^{i} \sum_{1\le j\le r} f^{j}\left(\frac{1}{\gamma^i}-\frac{1}{\gamma^k}\right) - \sum_{1\le k\le r} p_3^{k} \sum_{1\le i\le r} p_2^{i} \sum_{1\le j\le r} p_1^{j}\left(\frac{1}{\gamma^i}-\frac{1}{\gamma^k}\right)$$

$$\Delta_{31} = \sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_1} f^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\varepsilon^k}\right)$$

$$-\sum_{1\le k\le m_{13}} p_3^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{12}} p_2^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_1} p_1^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\varepsilon^k}\right)$$

$$\Delta_{32} = \sum_{1\leq k\leq m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\leq i\leq r} f^i \sum_{1\leq j\leq m_1} f^{jr+} \left(\frac{1}{\gamma} - \frac{1}{\varepsilon^k}\right)$$

$$- \sum_{1\leq k\leq m_{13}} p_3^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\leq i\leq r} p_2^i \sum_{1\leq j\leq m_1} p_1^{r+j} \left(\frac{1}{\gamma} - \frac{1}{\varepsilon^k}\right)$$

$$\Delta_{33} = \sum_{1\leq k\leq m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\leq i\leq m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\leq j\leq m_1} f^{r+j} \left(\frac{1}{\gamma} - \frac{1}{\delta^k}\right)$$

$$- \sum_{1\leq k\leq m_{23}} p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\leq i\leq m_{12}} p_2^{r+m_1+m_2+m_3+m_{12}+i} \sum_{1\leq j\leq m_1} p_1^{r+j} \left(\frac{1}{\gamma} - \frac{1}{\delta^k}\right)$$

$$\Delta_{34} = \sum_{1\leq k\leq m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\leq i\leq m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\leq j\leq m_1} f^{r+j} \left(\frac{1}{\gamma} - \frac{1}{\delta^k}\right)$$

$$- \sum_{1\leq k\leq m_{23}} p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\leq i\leq m_{23}} p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\leq j\leq m_1} p_1^{r+j} \left(\frac{1}{\gamma} - \frac{1}{\delta^k}\right)$$

$$\Delta_{35} = \sum_{1\leq k\leq m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\leq i\leq r} f^i \sum_{1\leq j\leq m_1} f^{r+j} \left(\frac{1}{\gamma} - \frac{1}{\delta^k}\right)$$

$$- \sum_{1\leq k\leq m_{23}} p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\leq i\leq r} p_2^i \sum_{1\leq j\leq m_1} p_1^{r+j} \left(\frac{1}{\gamma} - \frac{1}{\delta^k}\right)$$

$$\Delta_{36} = \sum_{1\leq k\leq r} f^k \sum_{1\leq i\leq r} f^i \sum_{1\leq j\leq m_1} f^{r+j} \left(\frac{1}{\gamma} - \frac{1}{\gamma^k}\right) - \sum_{1\leq k\leq r} p_3^k \sum_{1\leq i\leq r} p_2^i \sum_{1\leq j\leq m_1} p_1^{r+j} \left(\frac{1}{\gamma} - \frac{1}{\gamma^k}\right)$$

$$\Delta_{37} = \sum_{1\leq k\leq r} f^k \sum_{1\leq i\leq r} f^i \sum_{1\leq j\leq m_{12}} f^{r+m_1+m_2+m_3+j} \left(\frac{1}{\zeta^j} - \frac{1}{\gamma^k}\right)$$

$$- \sum_{1\leq k\leq r} p_3^k \sum_{1\leq i\leq r} p_2^i \sum_{1\leq j\leq m_{12}} p_1^{r+m_1+m_2+m_3+j} \left(\frac{1}{\zeta^j} - \frac{1}{\gamma^k}\right)$$

$$\Delta_{38} = \sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\varepsilon^j}-\frac{1}{\varepsilon^k}\right)$$

$$-\sum_{1\le k\le m_{13}} p_3^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{12}} p_2^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_{13}} p_1^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\varepsilon^j}-\frac{1}{\varepsilon^k}\right)$$

$$\Delta_{39} = \sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le r} f^i \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\varepsilon^j}-\frac{1}{\varepsilon^k}\right)$$

$$-\sum_{1\le k\le m_{13}} p_3^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le r} p_2^i \sum_{1\le j\le m_{13}} p_1^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\varepsilon^j}-\frac{1}{\varepsilon^k}\right)$$

$$\Delta_{40} = \sum_{1\le k\le r} f^k \sum_{1\le i\le r} f^i \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\varepsilon^j}-\frac{1}{\gamma^k}\right)$$

$$-\sum_{1\le k\le r} p_3^k \sum_{1\le i\le r} p_2^i \sum_{1\le j\le m_{13}} p_1^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\varepsilon^j}-\frac{1}{\gamma^k}\right)$$

$$\Delta_{41} = \sum_{1\le k\le r} f^k \sum_{1\le i\le r} f^i \sum_{1\le j\le r} f^j\left(\frac{1}{\gamma^j}-\frac{1}{\gamma^k}\right)-\sum_{1\le k\le r} p_3^k \sum_{1\le i\le r} p_2^i \sum_{1\le j\le r} p_1^j\left(\frac{1}{\gamma^j}-\frac{1}{\gamma^k}\right)$$

$$\Delta_{42} = \sum_{1\le k\le r} f^k \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_1} f^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\gamma^k}\right)$$

$$-\sum_{1\le k\le r} p_3^k \sum_{1\le i\le m_{12}} p_2^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_1} p_1^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\gamma^\kappa}\right)$$

$$\Delta_{43} = \sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_1} f^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\varepsilon^k}\right)$$

$$-\sum_{1\le k\le m_{13}} p_3^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{23}} p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_1} p_1^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\varepsilon^k}\right)$$

$$\varDelta_{44}=\sum_{1\le k\le m_{23}}f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k}\sum_{1\le i\le m_{23}}f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i}\sum_{1\le j\le m_1}f^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\delta^k}\right)$$

$$-\sum_{1\le k\le m_{23}}p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k}\sum_{1\le i\le m_{23}}p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i}\sum_{1\le j\le m_1}p_1^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\delta^k}\right)$$

$$\varDelta_{45}=\sum_{1\le k\le r}f^k\sum_{1\le i\le m_{23}}f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i}\sum_{1\le j\le m_1}f^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\gamma^k}\right)$$

$$-\sum_{1\le k\le r}p_3^k\sum_{1\le i\le m_{23}}p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i}\sum_{1\le j\le m_1}p_1^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\gamma^k}\right)$$

$$\varDelta_{46}=\sum_{1\le k\le r}f^k\sum_{1\le i\le r}f^i\sum_{1\le j\le m_1}f^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\gamma^k}\right)-\sum_{1\le k\le r}p_3^k\sum_{1\le i\le r}p_2^i\sum_{1\le j\le m_1}p_1^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\gamma^k}\right)$$

$$\varDelta_{47}=\sum_{1\le k\le r}f^k\sum_{1\le i\le m_{12}}f^{r+m_1+m_2+m_3+i}\sum_{1\le j\le m_{12}}f^{r+m_1+m_2+m_3+j}\left(\frac{1}{\zeta^j}-\frac{1}{\gamma^k}\right)$$

$$-\sum_{1\le k\le r}p_3^k\sum_{1\le i\le m_{12}}p_2^{r+m_1+m_2+m_3+i}\sum_{1\le j\le m_{12}}p_1^{r+m_1+m_2+m_3+j}\left(\frac{1}{\zeta^j}-\frac{1}{\gamma^k}\right)$$

$$\varDelta_{48}=\sum_{1\le k\le r}f^k\sum_{1\le i\le r}f^i\sum_{1\le j\le m_{12}}f^{r+m_1+m_2+m_3+j}\left(\frac{1}{\zeta^j}-\frac{1}{\gamma^k}\right)$$

$$-\sum_{1\le k\le r}p_3^k\sum_{1\le i\le r}p_2^i\sum_{1\le j\le m_{12}}p_1^{r+m_1+m_2+m_3+j}\left(\frac{1}{\zeta^j}-\frac{1}{\gamma^k}\right)$$

$$\varDelta_{49}=\sum_{1\le k\le r}f^k\sum_{1\le i\le m_{12}}f^{r+m_1+m_2+m_3+i}\sum_{1\le j\le m_{13}}f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\varepsilon^j}-\frac{1}{\gamma^k}\right)$$

$$-\sum_{1\le k\le r}p_3^k\sum_{1\le i\le m_{12}}p_2^{r+m_1+m_2+m_3+i}\sum_{1\le j\le m_{13}}p_1^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\varepsilon^j}-\frac{1}{\gamma^k}\right)$$

$$\Delta_{50} = \sum_{1\le k\le r} f^k \sum_{1\le i\le r} f^i \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\varepsilon^j}-\frac{1}{\gamma^k}\right)$$
$$-\sum_{1\le k\le r} p_3^k \sum_{1\le i\le r} p_2^i \sum_{1\le j\le m_{13}} p_1^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\varepsilon^j}-\frac{1}{\gamma^k}\right)$$

$$\Delta_{51} = \sum_{1\le k\le r} f^k \sum_{1\le i\le r} f^i \sum_{1\le j\le r} f^j\left(\frac{1}{\gamma^j}-\frac{1}{\gamma^k}\right)-\sum_{1\le k\le r} p_3^k \sum_{1\le i\le r} p_2^i \sum_{1\le j\le r} p_1^j\left(\frac{1}{\gamma^j}-\frac{1}{\gamma^k}\right)$$

$$\Delta_{52} = \sum_{1\le k\le m_3} f^{r+m_1+m_2+k} \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} f^{r+m_1+m_2+m_3+i}\left(\frac{2}{\zeta^i}-\frac{1}{a}\right)$$
$$-\sum_{1\le k\le m_3} p_3^{r+m_1+m_2+k} \sum_{1\le i\le m_{12}} p_2^{r+m_1+m_2+m_3+i} p_1^{r+m_1+m_2+m_3+i}\left(\frac{2}{\zeta^i}-\frac{1}{a}\right)$$

$$\Delta_{53} = \sum_{1\le k\le m_3} f^{r+m_1+m_2+k} \sum_{1\le i\le r} f^i f^i\left(\frac{2}{\gamma^i}-\frac{1}{a}\right)-\sum_{1\le k\le m_3} p_3^{r+m_1+m_2+k} \sum_{1\le i\le r} p_2^i p_1^i\left(\frac{2}{\gamma^i}-\frac{1}{a}\right)$$

$$\Delta_{54} = \sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} f^{r+m_1+m_2+m_3+i}\left(\frac{2}{\zeta^i}-\frac{1}{\varepsilon^k}\right)$$
$$-\sum_{1\le k\le m_{13}} p_3^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{12}} p_2^{r+m_1+m_2+m_3+i} p_1^{r+m_1+m_2+m_3+i}\left(\frac{2}{\zeta^i}-\frac{1}{\varepsilon^k}\right)$$

$$\Delta_{55} = \sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le r} f^i f^i\left(\frac{2}{\gamma^i}-\frac{1}{\varepsilon^k}\right)$$
$$-\sum_{1\le k\le m_{13}} p_3^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le r} p_2^i p_1^i\left(\frac{2}{\gamma^i}-\frac{1}{\varepsilon^k}\right)$$

$$\Delta_{56} = \sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} f^{r+m_1+m_2+m_3+i}\left(\frac{2}{\zeta^i}-\frac{1}{\delta^k}\right)$$

$$-\sum_{1\leq k\leq m_{23}} p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\leq i\leq m_{12}} p_2^{r+m_1+m_2+m_3+i} p_1^{r+m_1+m_2+m_3+i}\left(\frac{2}{\zeta^i}-\frac{1}{\delta^k}\right)$$

$$\Delta_{57}=\sum_{1\leq k\leq m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\leq i\leq r} f^i f^i\left(\frac{2}{\gamma^i}-\frac{1}{\delta^k}\right)$$
$$-\sum_{1\leq k\leq m_{23}} p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\leq i\leq m_{12}} p_2^i p_1^i\left(\frac{2}{\gamma^i}-\frac{1}{\delta^k}\right)$$

$$\Delta_{58}=\sum_{1\leq k\leq r} f^k \sum_{1\leq i\leq r} f^i f^i\left(\frac{2}{\gamma^i}-\frac{1}{\gamma^k}\right)-\sum_{1\leq k\leq r} p_3^k \sum_{1\leq i\leq r} p_2^i p_1^i\left(\frac{2}{\gamma^i}-\frac{1}{\gamma^k}\right)$$

$$\Delta_{59}=\sum_{1\leq k\leq r} f^k \sum_{1\leq i\leq m_{12}} f^{r+m_1+m_2+m_3+i} f^{r+m_1+m_2+m_3+i}\left(\frac{2}{\zeta^i}-\frac{1}{\gamma^k}\right)$$
$$-\sum_{1\leq k\leq r} p_3^k \sum_{1\leq i\leq m_{12}} p_2^{r+m_1+m_2+m_3+i} p_1^{r+m_1+m_2+m_3+i}\left(\frac{2}{\zeta^i}-\frac{1}{\gamma^k}\right)$$

$$\Delta_{60}=\sum_{1\leq k\leq r} f^k \sum_{1\leq i\leq r} f^i f^i\left(\frac{2}{\zeta^i}-\frac{1}{\delta^k}\right)-\sum_{1\leq k\leq r} p_3^k \sum_{1\leq i\leq r} p_2^i p_1^i\left(\frac{2}{\zeta^i}-\frac{1}{\delta^k}\right)$$

Οι υπόλοιπες 243 διαφορές που απομένουν από την Πλήρως Μικτή Ισορροπία Nash είναι θετικές. Σύμφωνα με την Πρόταση 3.3 και την Πρόταση 3.4 στο Παράρτημα Α είναι εύκολο να δούμε ότι οι διαφορές $\Delta_{62}+\cdots+\Delta_{303}$ είναι θετικές. Αυτό που μένει να κάνουμε είναι να προσθέσουμε το άθροισμα αυτών των διαφορών με το άθροισμα $\Delta_2+\cdots+\Delta_{60}$ και να δείξουμε ότι η πρόσθεση αυτή δίνει θετικό αποτέλεσμα.
Στο επόμενο υποκεφάλαιο θα δούμε μια μερική απλοποίηση μερικών από αυτές τις διαφορές έτσι ώστε να εκτελέσουμε την πρόσθεση μεταξύ τους.

8.2 Απλοποίηση και κάτω όρια

Θα απλοποιήσουμε ξεχωριστά και θα δώσουμε κάτω όρια στις διαφορές $\Delta_1, \Delta_2, \ldots \Delta_{60}$:

1. Απλοποίηση της Δ_1

$$\Delta_1 = IC_3^1(F) - IC_3^1(P)$$

$$= \frac{1 + 2f^1}{\gamma^1} - \frac{1 + p_1^1 + p_2^1}{\gamma^1}$$

$$= \frac{1 + 2f^1 - 1 - p_1^1 - p_2^1}{\gamma^1}$$

$$= \frac{2f^1 - p_1^1 - p_2^1}{\gamma^1}$$

$$= \frac{f^1 - p_1^1 + f^1 - p_2^1}{\gamma^1}$$

$$= \frac{\frac{(m+2)\gamma^1}{2C} - \frac{1}{2} - \frac{\gamma^1}{2\beta} - \frac{\gamma^1}{2\alpha} + \frac{\gamma^1}{2\gamma} + \frac{1}{2} + \frac{(m+2)\gamma^1}{2C} - \frac{1}{2} - \frac{\gamma^1}{2\alpha} + \frac{\gamma^1}{2\beta} - \frac{\gamma^1}{2\gamma} + \frac{1}{2}}{\gamma^1}$$

$$= \frac{\frac{(m+2)\gamma^1}{2C} - \frac{\gamma^1}{2\alpha} + \frac{(m+2)\gamma^1}{2C} - \frac{\gamma^1}{2\alpha}}{\gamma^1} = \frac{(m+2)}{C} - \frac{1}{\alpha} = A$$

2. Απλοποίηση της Δ_2

$$\Delta_2 = \sum_{1 \le k \le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1 \le i \le m_2} f^{r+m_1+i} \sum_{1 \le j \le m_1} f^{r+j} \left(\frac{1}{\beta} - \frac{1}{\delta^k}\right)$$
$$- \sum_{1 \le k \le m_{23}} p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1 \le i \le m_2} p_2^{r+m_1+i} \sum_{1 \le j \le m_1} p_1^{r+j} \left(\frac{1}{\beta} - \frac{1}{\delta^k}\right) =$$

$$= \sum_{1 \le k \le m_{23}} \left(f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} m_2 f^{r+m_1+1} m_1 f^{r+1} \right.$$
$$\left. - p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1 \le i \le m_2} p_2^{r+m_1+i} \sum_{1 \le j \le m_1} p_1^{r+j} \right) \left(\frac{1}{\beta} - \frac{1}{\delta^k}\right)$$

$$\Delta_2 = \sum_{1\le k\le m_{23}} \Bigg(f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} m_2 f^{r+m_1+1} m_1 f^{r+1} - p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \left(\frac{(r+2)\alpha-\Gamma}{2\alpha} - \frac{Z-m_{12}\gamma}{\gamma} - \frac{\Delta-m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma-\beta)}{2\beta\gamma} \right) \left(\frac{(r+2)\beta-\Gamma}{2\beta} - \frac{Z-m_{12}\beta}{\beta} - \frac{E-m_{13}\alpha}{\alpha} + \frac{\Gamma(\alpha-\gamma)}{2\alpha\gamma} \right) \Bigg) \left(\frac{1}{\beta} - \frac{1}{\delta^k} \right)$$

$$= \sum_{1\le k\le m_{23}} \left(\frac{1}{2}\left(\delta^k B + \frac{\delta^k}{\beta} - 1\right) m_1 m_2 \beta B \gamma G - \left(\frac{\delta^k}{\beta} - 1\right) \left(\frac{(r+2)\alpha-\Gamma}{2\alpha} - \frac{Z-m_{12}\gamma}{\gamma} - \frac{\Delta-m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma-\beta)}{2\beta\gamma} \right) \left(\frac{(r+2)\beta-\Gamma}{2\beta} - \frac{Z-m_{12}\beta}{\beta} - \frac{E-m_{13}\alpha}{\alpha} + \frac{\Gamma(\alpha-\gamma)}{2\alpha\gamma} \right) \right) \left(\frac{1}{\beta} - \frac{1}{\delta^k} \right)$$

$$= \sum_{1\le k\le m_{23}} \left(\left(\delta^k B + \frac{\delta^k}{\beta} - 1\right) \left(\frac{1}{\beta} - \frac{1}{\delta^k} \right) \frac{m_1 m_2 \beta B \gamma G}{2} - \left(\frac{\delta^k}{\beta} - 1\right) \left(\frac{(r+2)\alpha-\Gamma}{2\alpha} - \frac{Z-m_{12}\gamma}{\gamma} - \frac{\Delta-m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma-\beta)}{2\beta\gamma} \right) \left(\frac{(r+2)\beta-\Gamma}{2\beta} - \frac{Z-m_{12}\beta}{\beta} - \frac{E-m_{13}\alpha}{\alpha} + \frac{\Gamma(\alpha-\gamma)}{2\alpha\gamma} \right) \left(\frac{1}{\beta} - \frac{1}{\delta^k} \right) \right) =$$

$$= \frac{m_1 m_2 \beta B \gamma G}{2} \sum_{1\le k\le m_{23}} \left(\left(\delta^k B + \frac{\delta^k}{\beta} - 1\right) \left(\frac{1}{\beta} - \frac{1}{\delta^k} \right) \right) - \left(\frac{(r+2)\alpha-\Gamma}{2\alpha} - \frac{Z-m_{12}\gamma}{\gamma} - \frac{\Delta-m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma-\beta)}{2\beta\gamma} \right) \left(\frac{(r+2)\beta-\Gamma}{2\beta} - \frac{Z-m_{12}\beta}{\beta} - \frac{E-m_{13}\alpha}{\alpha} + \frac{\Gamma(\alpha-\gamma)}{2\alpha\gamma} \right) \sum_{1\le k\le m_{23}} \left(\left(\frac{\delta^k}{\beta} - 1\right) \left(\frac{1}{\beta} - \frac{1}{\delta^k} \right) \right)$$

$$= \frac{m_1 m_2 \beta B^2 \gamma G}{2} \sum_{1\le k\le m_{23}} \delta^k \left(\frac{1}{\beta} - \frac{1}{\delta^k} \right) + \frac{m_1 m_2 \beta B \gamma G}{2} \sum_{1\le k\le m_{23}} \left(\left(\frac{\delta^k}{\beta} - 1\right) \left(\frac{1}{\beta} - \frac{1}{\delta^k} \right) \right) - \left(\frac{(r+2)\alpha-\Gamma}{2\alpha} - \frac{Z-m_{12}\gamma}{\gamma} - \frac{\Delta-m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma-\beta)}{2\beta\gamma} \right) \left(\frac{(r+2)\beta-\Gamma}{2\beta} - \frac{Z-m_{12}\beta}{\beta} - \frac{E-m_{13}\alpha}{\alpha} + \frac{\Gamma(\alpha-\gamma)}{2\alpha\gamma} \right) \sum_{1\le k\le m_{23}} \left(\left(\frac{\delta^k}{\beta} - 1\right) \left(\frac{1}{\beta} - \frac{1}{\delta^k} \right) \right)$$

$= \frac{m_1 m_2 \beta B^2 \gamma G}{2} \sum_{1\le k\le m_{23}} \left(\frac{\delta^k-\beta}{\beta}\right) + \left(\frac{m_1 m_2 \beta B \gamma G}{2} - \left(\frac{(r+2)\alpha-\Gamma}{2\alpha} - \frac{Z-m_{12}\gamma}{\gamma} - \frac{\Delta-m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\left(\frac{(r+2)\beta-\Gamma}{2\beta} - \frac{Z-m_{12}\beta}{\beta} - \frac{E-m_{13}\alpha}{\alpha} + \frac{\Gamma(\alpha-\gamma)}{2\alpha\gamma}\right)\right) \sum_{1\le k\le m_{23}} \left(\left(\frac{\delta^k}{\beta}-1\right)\left(\frac{1}{\beta}-\frac{1}{\delta^k}\right)\right)$

$= \frac{m_1 m_2 \beta B^2 \gamma G}{2} \left(\frac{\Delta-m_{23}\beta}{\beta}\right) + \left(\frac{m_1 m_2 \beta B \gamma G}{2} - \left(\frac{(r+2)\alpha-\Gamma}{2\alpha} - \frac{Z-m_{12}\gamma}{\gamma} - \frac{\Delta-m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\left(\frac{(r+2)\beta-\Gamma}{2\beta} - \frac{Z-m_{12}\beta}{\beta} - \frac{E-m_{13}\alpha}{\alpha} + \frac{\Gamma(\alpha-\gamma)}{2\alpha\gamma}\right)\right) \sum_{1\le k\le m_{23}} \left(\left(\frac{\delta^k-\beta}{\beta}\right)\left(\frac{\delta^k-\beta}{\delta^k\beta}\right)\right)$

$= \frac{m_1 m_2 \beta B^2 \gamma G}{2} \left(\frac{\Delta-m_{23}\beta}{\beta}\right) + \left(\frac{m_1 m_2 \beta B \gamma G}{2} - \left(\frac{(r+2)\alpha-\Gamma}{2\alpha} - \frac{Z-m_{12}\gamma}{\gamma} - \frac{\Delta-m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\left(\frac{(r+2)\beta-\Gamma}{2\beta} - \frac{Z-m_{12}\beta}{\beta} - \frac{E-m_{13}\alpha}{\alpha} + \frac{\Gamma(\alpha-\gamma)}{2\alpha\gamma}\right)\right) \sum_{1\le k\le m_{23}} \left(\frac{1}{\delta^k}\left(\frac{\delta^k-\beta}{\beta}\right)^2\right)$

$= \frac{m_1 m_2 \beta B^2 \gamma G}{2} \left(\frac{\Delta-m_{23}\beta}{\beta}\right) + \left(\frac{m_1 m_2 \beta B \gamma G}{2} - \left(\frac{(r+2)\alpha-\Gamma}{2\alpha} - \frac{Z-m_{12}\gamma}{\gamma} - \frac{\Delta-m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\left(\frac{(r+2)\beta-\Gamma}{2\beta} - \frac{Z-m_{12}\beta}{\beta} - \frac{E-m_{13}\alpha}{\alpha} + \frac{\Gamma(\alpha-\gamma)}{2\alpha\gamma}\right)\right) \frac{1}{\beta^2} \sum_{1\le k\le m_{23}} \left(\frac{\left(\delta^k-\beta\right)^2}{\delta^k}\right)$

$\ge \frac{m_1 m_2 \beta B^2 \gamma G}{2} \left(\frac{\Delta-m_{23}\beta}{\beta}\right) + \left(\frac{m_1 m_2 \beta B \gamma G}{2} - \left(\frac{(r+2)\alpha-\Gamma}{2\alpha} - \frac{Z-m_{12}\gamma}{\gamma} - \frac{\Delta-m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\left(\frac{(r+2)\beta-\Gamma}{2\beta} - \frac{Z-m_{12}\beta}{\beta} - \frac{E-m_{13}\alpha}{\alpha} + \frac{\Gamma(\alpha-\gamma)}{2\alpha\gamma}\right)\right) \frac{1}{\beta^2} \sum_{1\le k\le m_{23}} \left(\frac{\left(\delta^k-\gamma\right)^2}{\delta^k}\right)$

(Αφού $\gamma \ge \beta$)

3. Απλοποίηση της Δ_3

$$\Delta 3 = \sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le m_1} f^{r+j}\left(\frac{1}{\beta}-\frac{1}{\varepsilon^k}\right) - \sum_{1\le k\le m_{13}} p_3^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_2} p_2^{r+m_1+i} \sum_{1\le j\le m_1} p_1^{r+j}\left(\frac{1}{\beta}-\frac{1}{\varepsilon^k}\right)$$

$= \sum_{1\le k\le m_{23}} \left(f^{r+m_1+m_2+m_3+m_{12}+k} m_2 f^{r+m_1+1} m_1 f^{r+1} - p_3^{r+m_1+m_2+m_3+m_{12}+k}\left(\frac{(r+2)\alpha-\Gamma}{2\alpha} - \frac{Z-m_{12}\gamma}{\gamma} - \frac{\Delta-m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\left(\frac{(r+2)\beta-\Gamma}{2\beta} - \frac{Z-m_{12}\beta}{\beta} - \frac{E-m_{13}\alpha}{\alpha} + \frac{\Gamma(\alpha-\gamma)}{2\alpha\gamma}\right)\right)\left(\frac{1}{\beta}-\frac{1}{\delta^k}\right)$

$$= \sum_{1\le k\le m_{13}} \left(\frac{1}{2}\left(G\varepsilon^k + \frac{\varepsilon^k}{\gamma} - 1\right) m_1 m_2 \beta B \gamma G - \left(\frac{\varepsilon^k}{\gamma} - 1\right)\left(\frac{(r+2)\alpha-\Gamma}{2\alpha} - \frac{Z-m_{12}\gamma}{\gamma} - \frac{\Delta - m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\left(\frac{(r+2)\beta-\Gamma}{2\beta} - \frac{Z-m_{12}\beta}{\beta} - \frac{E-m_{13}\alpha}{\alpha} + \frac{\Gamma(\alpha-\gamma)}{2\alpha\gamma}\right)\right)\left(\frac{1}{\beta} - \frac{1}{\varepsilon^k}\right)$$

$$= \sum_{1\le k\le m_{13}} \left(\frac{m_1 m_2 \beta B \gamma G}{2}\left(G\varepsilon^k + \frac{\varepsilon^k}{\gamma} - 1\right)\left(\frac{1}{\beta} - \frac{1}{\varepsilon^k}\right) - \left(\frac{\varepsilon^k}{\gamma} - 1\right)\left(\frac{(r+2)\alpha-\Gamma}{2\alpha} - \frac{Z-m_{12}\gamma}{\gamma} - \frac{\Delta - m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\left(\frac{(r+2)\beta-\Gamma}{2\beta} - \frac{Z-m_{12}\beta}{\beta} - \frac{E-m_{13}\alpha}{\alpha} + \frac{\Gamma(\alpha-\gamma)}{2\alpha\gamma}\right)\left(\frac{1}{\beta} - \frac{1}{\varepsilon^k}\right)\right)$$

$$= \frac{m_1 m_2 \beta B \gamma G}{2} \sum_{1\le k\le m_{13}} \left(\left(G\varepsilon^k + \frac{\varepsilon^k}{\gamma} - 1\right)\left(\frac{1}{\beta} - \frac{1}{\varepsilon^k}\right) \right) - \left(\frac{(r+2)\alpha-\Gamma}{2\alpha} - \frac{Z-m_{12}\gamma}{\gamma} - \frac{\Delta - m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\left(\frac{(r+2)\beta-\Gamma}{2\beta} - \frac{Z-m_{12}\beta}{\beta} - \frac{E-m_{13}\alpha}{\alpha} + \frac{\Gamma(\alpha-\gamma)}{2\alpha\gamma}\right)\sum_{1\le k\le m_{13}} \left(\left(\frac{\varepsilon^k}{\gamma} - 1\right)\left(\frac{1}{\beta} - \frac{1}{\varepsilon^k}\right)\right)$$

$$= \frac{m_1 m_2 \beta G B \gamma G}{2} \sum_{1\le k\le m_{13}} \varepsilon^k \left(\frac{1}{\beta} - \frac{1}{\varepsilon^k}\right) + \frac{m_1 m_2 \beta B \gamma G}{2} \sum_{1\le k\le m_{13}} \left(\left(\frac{\varepsilon^k}{\gamma} - 1\right)\left(\frac{1}{\beta} - \frac{1}{\varepsilon^k}\right)\right) - \left(\frac{(r+2)\alpha-\Gamma}{2\alpha} - \frac{Z-m_{12}\gamma}{\gamma} - \frac{\Delta - m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\left(\frac{(r+2)\beta-\Gamma}{2\beta} - \frac{Z-m_{12}\beta}{\beta} - \frac{E-m_{13}\alpha}{\alpha} + \frac{\Gamma(\alpha-\gamma)}{2\alpha\gamma}\right)\sum_{1\le k\le m_{13}} \left(\left(\frac{\varepsilon^k}{\gamma} - 1\right)\left(\frac{1}{\beta} - \frac{1}{\varepsilon^k}\right)\right)$$

$$= \frac{m_1 m_2 \beta G B \gamma G}{2} \sum_{1\le k\le m_{13}} \varepsilon^k \left(\frac{1}{\beta} - \frac{1}{\varepsilon^k}\right) + \frac{m_1 m_2 \beta B \gamma G}{2} \sum_{1\le k\le m_{13}} \left(\left(\frac{\varepsilon^k}{\gamma} - 1\right)\left(\frac{1}{\beta} - \frac{1}{\varepsilon^k}\right)\right) - \left(\frac{(r+2)\alpha-\Gamma}{2\alpha} - \frac{Z-m_{12}\gamma}{\gamma} - \frac{\Delta - m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\left(\frac{(r+2)\beta-\Gamma}{2\beta} - \frac{Z-m_{12}\beta}{\beta} - \frac{E-m_{13}\alpha}{\alpha} + \frac{\Gamma(\alpha-\gamma)}{2\alpha\gamma}\right)\sum_{1\le k\le m_{13}} \left(\left(\frac{\varepsilon^k}{\gamma} - 1\right)\left(\frac{1}{\beta} - \frac{1}{\varepsilon^k}\right)\right)$$

$$\frac{m_1 m_2 \beta G B \gamma G}{2} \sum_{1\le k\le m_{13}} \left(\frac{\varepsilon^k - \beta}{\beta}\right) + \left(\frac{m_1 m_2 \beta B \gamma G}{2} - \left(\frac{(r+2)\alpha-\Gamma}{2\alpha} - \frac{Z-m_{12}\gamma}{\gamma} - \frac{\Delta - m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\left(\frac{(r+2)\beta-\Gamma}{2\beta} - \frac{Z-m_{12}\beta}{\beta} - \frac{E-m_{13}\alpha}{\alpha} + \frac{\Gamma(\alpha-\gamma)}{2\alpha\gamma}\right)\right)\sum_{1\le k\le m_{13}} \left(\left(\frac{\varepsilon^k}{\gamma} - 1\right)\left(\frac{1}{\beta} - \frac{1}{\varepsilon^k}\right)\right)$$

$$= \frac{m_1 m_2 \beta G B \gamma G}{2} \left(\frac{E - m_{13}\beta}{\beta}\right) + \left(\frac{m_1 m_2 \beta B \gamma G}{2} - \left(\frac{(r+2)\alpha-\Gamma}{2\alpha} - \frac{Z-m_{12}\gamma}{\gamma} - \frac{\Delta - m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\left(\frac{(r+2)\beta-\Gamma}{2\beta} - \frac{Z-m_{12}\beta}{\beta} - \frac{E-m_{13}\alpha}{\alpha} + \frac{\Gamma(\alpha-\gamma)}{2\alpha\gamma}\right)\right)\sum_{1\le k\le m_{13}} \left(\left(\frac{\varepsilon^k - \gamma}{\gamma}\right)\left(\frac{\varepsilon^k - \beta}{\varepsilon^k \beta}\right)\right)$$

$= \frac{m_1 m_2 \beta G B \gamma G}{2}\left(\frac{E-m_{13}\beta}{\beta}\right) + \left(\frac{m_1 m_2 \beta B \gamma G}{2} - \left(\frac{(r+2)\alpha-\Gamma}{2\alpha} - \frac{Z-m_{12}\gamma}{\gamma} - \frac{\Delta-m_{23}\alpha}{\alpha} + \right.\right.$

$\left.\left.\frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\left(\frac{(r+2)\beta-\Gamma}{2\beta} - \frac{Z-m_{12}\beta}{\beta} - \frac{E-m_{13}\alpha}{\alpha} + \frac{\Gamma(\alpha-\gamma)}{2\alpha\gamma}\right)\right)\frac{1}{\beta\gamma}\sum_{1\le k\le m_{13}}\left((\varepsilon^k-\gamma)\left(\frac{\varepsilon^k-\beta}{\varepsilon^k}\right)\right)$

$\ge \frac{m_1 m_2 \beta G B \gamma G}{2}\left(\frac{E-m_{13}\beta}{\beta}\right) + \left(\frac{m_1 m_2 \beta B \gamma G}{2} - \left(\frac{(r+2)\alpha-\Gamma}{2\alpha} - \frac{Z-m_{12}\gamma}{\gamma} - \frac{\Delta-m_{23}\alpha}{\alpha} + \right.\right.$

$\left.\left.\frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\left(\frac{(r+2)\beta-\Gamma}{2\beta} - \frac{Z-m_{12}\beta}{\beta} - \frac{E-m_{13}\alpha}{\alpha} + \frac{\Gamma(\alpha-\gamma)}{2\alpha\gamma}\right)\right)\frac{1}{\beta\gamma}\sum_{1\le k\le m_{13}}\left(\frac{(\varepsilon^k-\gamma)^2}{\varepsilon^k}\right)$

(Αφού $\gamma \ge \beta$)

4. <u>Απλοποίηση της Δ_4</u>

$$\Delta 4 = \sum_{1\le k\le r} f^k \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le m_1} f^{r+j}\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right) - \sum_{1\le k\le r} p_3^k \sum_{1\le i\le m_2} p_2^{r+m_1+i} \sum_{1\le j\le m_1} p_1^{r+j}\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)$$

$$= \sum_{1\le k\le r} f^k m_2 f^{r+m_1+1} m_1 f^{r+1}\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right) - \sum_{1\le k\le r} p_3^k \sum_{1\le i\le m_2} p_2^{r+m_1+i} \sum_{1\le j\le m_1} p_1^{r+j}\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)$$

$$= \sum_{1\le k\le r}\left(f^k m_2 f^{r+m_1+1} m_1 f^{r+1} - p_3^k \sum_{1\le i\le m_2} p_2^{r+m_1+i} \sum_{1\le j\le m_1} p_1^{r+j}\right)\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)$$

$= \sum_{1\le k\le r}\left(f^k m_2 f^{r+m_1+1} m_1 f^{r+1} - p_3^k\left(\frac{(r+2)\alpha-\Gamma}{2\alpha} - \frac{Z-m_{12}\gamma}{\gamma} - \frac{\Delta-m_{23}\alpha}{\alpha} + \right.\right.$

$\left.\left.\frac{\Gamma(\gamma-\beta)}{2\beta}\right)\left(\frac{(r+2)\beta-\Gamma}{2\beta} - \frac{Z-m_{12}\beta}{\beta} - \frac{E-m_{13}\alpha}{\alpha} + \frac{\Gamma(\alpha-\gamma)}{2\beta}\right)\right)\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)$

$= \sum_{1\le k\le r}\left(\left(\gamma^k B + \frac{\gamma^k}{\beta} - 1\right)\frac{m_1 m_2 \beta B \gamma G}{2} - \left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2}\right)\left(\frac{(r+2)\alpha-\Gamma}{2\alpha} - \frac{Z-m_{12}\gamma}{\gamma} - \right.\right.$

$\left.\left.\frac{\Delta-m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\left(\frac{(r+2)\beta-\Gamma}{2\beta} - \frac{Z-m_{12}\beta}{\beta} - \frac{E-m_{13}\alpha}{\alpha} + \frac{\Gamma(\alpha-\gamma)}{2\alpha\gamma}\right)\right)\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)$

$= \sum_{1\le k\le r}\left(\left(\gamma^k B + \frac{\gamma^k}{\beta} - 1\right)\left(\frac{1}{\beta} - \frac{1}{\gamma^k}\right)\frac{m_1 m_2 \beta B \gamma G}{2} - \left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2}\right)\left(\frac{(r+2)\alpha - \Gamma}{2\alpha} -\right.\right.$
$\left.\left.\frac{Z - m_{12}\gamma}{\gamma} - \frac{\Delta - m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\left(\frac{(r+2)\beta - \Gamma}{2\beta} - \frac{Z - m_{12}\beta}{\beta} - \frac{E - m_{13}\alpha}{\alpha} + \frac{\Gamma(\alpha-\gamma)}{2\alpha\gamma}\right)\left(\frac{1}{\beta} - \frac{1}{\gamma^k}\right)\right)$

$= \frac{m_1 m_2 \beta B \gamma G}{2}\sum_{1\le k\le r}\left(\left(\gamma^k B\left(\frac{1}{\beta} - \frac{1}{\gamma^k}\right) + \left(\frac{\gamma^k}{\beta} - 1\right)\left(\frac{1}{\beta} - \frac{1}{\gamma^k}\right)\right)\right) - \left(\frac{(r+2)\alpha - \Gamma}{2\alpha} - \frac{Z - m_{12}\gamma}{\gamma} -\right.$
$\left.\frac{\Delta - m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\left(\frac{(r+2)\beta - \Gamma}{2\beta} - \frac{Z - m_{12}\beta}{\beta} - \frac{E - m_{13}\alpha}{\alpha} + \frac{\Gamma(\alpha-\gamma)}{2\alpha\gamma}\right)\sum_{1\le k\le r}\left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} -\right.$
$\left.\frac{1}{2}\right)\left(\frac{1}{\beta} - \frac{1}{\gamma^k}\right)$

$= \frac{m_1 m_2 \beta B^2 \gamma G}{2}\sum_{1\le k\le r}\left(\gamma^k\left(\frac{1}{\beta} - \frac{1}{\gamma^k}\right)\right) + \frac{m_1 m_2 \beta B \gamma G}{2}\sum_{1\le k\le r}\left(\left(\frac{\gamma^k}{\beta} - 1\right)\left(\frac{1}{\beta} - \frac{1}{\gamma^k}\right)\right) -$
$\left(\frac{(r+2)\alpha - \Gamma}{2\alpha} - \frac{Z - m_{12}\gamma}{\gamma} - \frac{\Delta - m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\left(\frac{(r+2)\beta - \Gamma}{2\beta} - \frac{Z - m_{12}\beta}{\beta} - \frac{E - m_{13}\alpha}{\alpha} +\right.$
$\left.\frac{\Gamma(\alpha-\gamma)}{2\alpha\gamma}\right)\sum_{1\le k\le r}\left(\left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2}\right)\left(\frac{1}{\beta} - \frac{1}{\gamma^k}\right)\right)$

Έχουμε ότι για $k \in [r]$,$p_1^k = \frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2}$ όπου $0 < p_1^k < 1$. Έτσι $p_1^k + p_3^k =$ $\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2} + \frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2} = \frac{\gamma^k}{\beta} - 1$. Επομένως, $\frac{\gamma^k}{\beta} - 1 \ge p_3^k$. Άρα,

$\ge \frac{m_1 m_2 \beta B^2 \gamma G}{2}\sum_{1\le k\le r}\left(\frac{\gamma^k - \beta}{\beta}\right) + \frac{m_1 m_2 \beta B \gamma G}{2}\sum_{1\le k\le r}\left(\left(\frac{\gamma^k}{\beta} - 1\right)\left(\frac{1}{\beta} - \frac{1}{\gamma^k}\right)\right) -$
$\left(\frac{(r+2)\alpha - \Gamma}{2\alpha} - \frac{Z - m_{12}\gamma}{\gamma} - \frac{\Delta - m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\left(\frac{(r+2)\beta - \Gamma}{2\beta} - \frac{Z - m_{12}\beta}{\beta} - \frac{E - m_{13}\alpha}{\alpha} +\right.$
$\left.\frac{\Gamma(\alpha-\gamma)}{2\alpha\gamma}\right)\sum_{1\le k\le r}\left(\left(\frac{\gamma^k}{\beta} - 1\right)\left(\frac{1}{\beta} - \frac{1}{\gamma^k}\right)\right)$

$=$
$\frac{m_1 m_2 \beta B^2 \gamma G}{2}\left(\frac{\gamma^k - \beta}{\beta}\right) +$
$\left(\frac{m_1 m_2 \beta B \gamma G}{2} - \left(\frac{(r+2)\alpha - \Gamma}{2\alpha} - \frac{Z - m_{12}\gamma}{\gamma} - \frac{\Delta - m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\left(\frac{(r+2)\beta - \Gamma}{2\beta} - \frac{Z - m_{12}\beta}{\beta} -\right.\right.$
$\left.\left.\frac{E - m_{13}\alpha}{\alpha} + \frac{\Gamma(\alpha-\gamma)}{2\alpha\gamma}\right)\right)\sum_{1\le k\le r}\left(\left(\frac{\gamma^k - \beta}{\beta}\right)\left(\frac{\gamma^k - \beta}{\gamma^k \beta}\right)\right)$

$=$
$\frac{m_1 m_2 \beta B^2 \gamma G}{2}\left(\frac{\Gamma - r\beta}{\beta}\right) +$
$\left(\frac{m_1 m_2 \beta B \gamma G}{2} - \left(\frac{(r+2)\alpha - \Gamma}{2\alpha} - \frac{Z - m_{12}\gamma}{\gamma} - \frac{\Delta - m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\left(\frac{(r+2)\beta - \Gamma}{2\beta} - \frac{Z - m_{12}\beta}{\beta} -\right.\right.$
$\left.\left.\frac{E - m_{13}\alpha}{\alpha} + \frac{\Gamma(\alpha-\gamma)}{2\alpha\gamma}\right)\right)\frac{1}{\beta^2}\sum_{1\le k\le r}\left(\frac{\left(\gamma^k - \beta\right)^2}{\gamma^k}\right)$

$\geq$
$\frac{m_1 m_2 \beta B^2 \gamma G}{2}\left(\frac{\Gamma - r\beta}{\beta}\right) +$
$\left(\frac{m_1 m_2 \beta B \gamma G}{2} - \left(\frac{(r+2)\alpha - \Gamma}{2\alpha} - \frac{Z - m_{12}\gamma}{\gamma} - \frac{\Delta - m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma - \beta)}{2\beta\gamma}\right)\left(\frac{(r+2)\beta - \Gamma}{2\beta} - \frac{Z - m_{12}\beta}{\beta} -\right.\right.$
$\left.\left.\frac{E - m_{13}\alpha}{\alpha} + \frac{\Gamma(\alpha - \gamma)}{2\alpha\gamma}\right)\right)\frac{1}{\beta^2}\sum_{1\leq k\leq r}\left(\frac{(\gamma^k - \gamma)^2}{\gamma^k}\right)$
(Αφού $\gamma \geq \beta$)

5. <u>Απλοποίηση της Δ_5</u>

$$\Delta 5 = \sum_{1\leq k\leq r} f^k \sum_{1\leq i\leq m_2} f^{r+m_1+i} \sum_{1\leq j\leq m_{12}} f^{r+m_1+m_2+m_3+j}\left(\frac{1}{\beta} - \frac{1}{\gamma^k}\right) - \sum_{1\leq k\leq r} p_3^k \sum_{1\leq i\leq m_2} p_2^{r+m_1+i} \sum_{1\leq j\leq m_{12}} p_1^{r+m_1+m_2+m_3+j}\left(\frac{1}{\beta} - \frac{1}{\gamma^k}\right) =$$

$$= \sum_{1\leq k\leq r}\sum_{1\leq i\leq m_2}\sum_{1\leq j\leq m_{12}} f^k f^{r+m_1+i} f^{r+m_1+m_2+m_3+j}\left(\frac{1}{\beta} - \frac{1}{\gamma^k}\right) - \sum_{1\leq k\leq r}\sum_{1\leq i\leq m_2}\sum_{1\leq j\leq m_{12}} p_3^k p_2^{r+m_1+i} p_1^{r+m_1+m_2+m_3+j}\left(\frac{1}{\beta} - \frac{1}{\gamma^k}\right)$$

$$= \sum_{1\leq k\leq r}\sum_{1\leq i\leq m_2}\sum_{1\leq j\leq m_{12}}\left(f^k f^{r+m_1+i} f^{r+m_1+m_2+m_3+j} - p_3^k p_2^{r+m_1+i} p_1^{r+m_1+m_2+m_3+j}\right)\left(\frac{1}{\beta} - \frac{1}{\gamma^k}\right)$$

$$= \sum_{1\leq k\leq r}\sum_{1\leq i\leq m_2}\sum_{1\leq j\leq m_{12}}\left(m_2 f^{r+m_1+1} f^k f^{r+m_1+m_2+m_3+j} - p_3^k p_2^{r+m_1+i} p_1^{r+m_1+m_2+m_3+j}\right)\left(\frac{1}{\beta} - \frac{1}{\gamma^k}\right)$$

$= \sum_{1\leq k\leq r}\sum_{1\leq j\leq m_{12}}\left(m_2 \beta B \frac{1}{2}\left(\gamma^k B + \frac{\gamma^k}{\beta} - 1\right)\frac{1}{2}\left(B\zeta^j + \frac{\zeta^j}{\beta} - 1\right) - \left(\frac{(r+2)\alpha - \Gamma}{2\alpha} -\right.\right.$
$\left.\left.\frac{Z - m_{12}\gamma}{\gamma} - \frac{\Delta - m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma - \beta)}{2\beta\gamma}\right)\left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2}\right)\left(\frac{\zeta^j}{\beta} - 1\right)\right)\left(\frac{1}{\beta} - \frac{1}{\gamma^k}\right)$

$= \sum_{1\leq k\leq r}\sum_{1\leq j\leq m_{12}}\left(m_2 \beta B \frac{1}{4}\left(\gamma^k B + \frac{\gamma^k}{\beta} - 1\right)\left(B\zeta^j + \frac{\zeta^j}{\beta} - 1\right)\left(\frac{1}{\beta} - \frac{1}{\gamma^k}\right) -\right.$
$\left.\left(\frac{(r+2)\alpha - \Gamma}{2\alpha} - \frac{Z - m_{12}\gamma}{\gamma} - \frac{\Delta - m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma - \beta)}{2\beta\gamma}\right)\left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2}\right)\left(\frac{\zeta^j}{\beta} - 1\right)\left(\frac{1}{\beta} - \frac{1}{\gamma^k}\right)\right)$

$$\geq \sum_{1\le k\le r}\sum_{1\le j\le m_{12}}\left(\frac{m_2\beta B}{4}\left(\frac{\gamma^k}{\beta}-1\right)\left(\frac{\zeta^j}{\beta}-1\right)\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)-\left(\frac{(r+2)\alpha-\Gamma}{2\alpha}-\frac{Z-m_{12}\gamma}{\gamma}-\frac{\Delta-m_{23}\alpha}{\alpha}+\frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\left(\frac{\gamma^k}{\beta}-1\right)\left(\frac{\zeta^j}{\beta}-1\right)\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)\right) \qquad \left(\text{Αφού}\,\frac{\gamma^k}{\beta}-1\geq\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)$$

$$=\sum_{1\le k\le r}\sum_{1\le j\le m_{12}}\left(\left(\frac{\gamma^k}{\beta}-1\right)\left(\frac{\zeta^j}{\beta}-1\right)\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)\left(\frac{\mathrm{m_2\beta B}}{4}-\left(\frac{(r+2)\alpha-\Gamma}{2\alpha}-\frac{Z-m_{12}\gamma}{\gamma}-\frac{\Delta-m_{23}\alpha}{\alpha}+\frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\right)\right)$$

$$=\left(\frac{\mathrm{m_2\beta B}}{4}-\left(\frac{(r+2)\alpha-\Gamma}{2\alpha}-\frac{Z-m_{12}\gamma}{\gamma}-\frac{\Delta-m_{23}\alpha}{\alpha}+\frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\right)\sum_{1\le k\le r}\sum_{1\le j\le m_{12}}\left(\left(\frac{\gamma^k-\beta}{\beta}\right)\left(\frac{\gamma^k-\beta}{\gamma^k\beta}\right)\left(\frac{\zeta^j-\beta}{\beta}\right)\right)=$$

$$=\left(\frac{\mathrm{m_2\beta B}}{4}-\left(\frac{(r+2)\alpha-\Gamma}{2\alpha}-\frac{Z-m_{12}\gamma}{\gamma}-\frac{\Delta-m_{23}\alpha}{\alpha}+\frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\right)\frac{1}{\beta^3}\sum_{1\le k\le r}\sum_{1\le j\le m_{12}}\left(\frac{\left(\gamma^k-\beta\right)^2}{\gamma^k}\left(\zeta^j-\beta\right)\right)$$

$$\geq\left(\frac{\mathrm{m_2\beta B}}{4}-\left(\frac{(r+2)\alpha-\Gamma}{2\alpha}-\frac{Z-m_{12}\gamma}{\gamma}-\frac{\Delta-m_{23}\alpha}{\alpha}+\frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\right)\frac{1}{\beta^3}\sum_{1\le k\le r}\sum_{1\le j\le m_{12}}\left(\frac{\left(\gamma^k-\gamma\right)^2}{\gamma^k}\left(\zeta^j-\beta\right)\right).$$

(Αφού $\gamma \geq \beta$)

6. Απλοποίηση της Δ_6

$$\Delta 6=\sum_{1\le k\le r} f^k \sum_{1\le i\le m_2} f^{r+m_1+i}\sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)$$
$$-\sum_{1\le k\le r} p_3^k \sum_{1\le i\le m_2} p_2^{r+m_1+i}\sum_{1\le j\le m_{13}} p_1^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)$$

$$=\sum_{1\le k\le r}\sum_{1\le i\le m_2}\sum_{1\le j\le m_{13}} f^k f^{r+m_1+i} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)$$
$$-\sum_{1\le k\le r}\sum_{1\le i\le m_2}\sum_{1\le j\le m_{13}} p_3^k p_2^{r+m_1+i} p_1^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)$$

$$=\sum_{1\le k\le r}\sum_{1\le i\le m_2}\sum_{1\le j\le m_{13}}\left(f^k f^{r+m_1+i} f^{r+m_1+m_2+m_3+m_{12}+j}-p_3^k p_2^{r+m_1+i} p_1^{r+m_1+m_2+m_3+m_{12}+j}\right)\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)$$

$$= \sum_{1\le k\le r}\sum_{1\le j\le m_{13}}\Bigg(f^k m_2 f^{r+m_1+1} f^{r+m_1+m_2+m_3+m_{12}+j} - p_3^k\Bigg(\frac{(r+2)\alpha-\Gamma}{2\alpha}-\frac{Z-m_{12}\gamma}{\gamma}-\frac{\Delta-m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\Bigg)p_1^{r+m_1+m_2+m_3+m_{12}+j}\Bigg)\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)$$

$$\sum_{1\le k\le r}\sum_{1\le j\le m_{13}}\Bigg(m_2\beta B\frac{1}{2}\left(\gamma^k B+\frac{\gamma^k}{\beta}-1\right)\frac{1}{2}\left(A\varepsilon^j+\frac{\varepsilon^j}{a}-1\right)-\left(\frac{(r+2)\alpha-\Gamma}{2\alpha}-\frac{Z-m_{12}\gamma}{\gamma}-\frac{\Delta-m_{23}\alpha}{\alpha}+\frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\varepsilon^j}{\alpha}-1\right)\Bigg)\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)$$

$$= \sum_{1\le k\le r}\sum_{1\le j\le m_{13}}\Bigg(m_2\beta B\frac{1}{4}\left(\gamma^k B+\frac{\gamma^k}{\beta}-1\right)\left(A\varepsilon^j+\frac{\varepsilon^j}{a}-1\right)\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)-\left(\frac{(r+2)\alpha-\Gamma}{2\alpha}-\frac{Z-m_{12}\gamma}{\gamma}-\frac{\Delta-m_{23}\alpha}{\alpha}+\frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\varepsilon^j}{\alpha}-1\right)\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)\Bigg)$$

$$= \sum_{1\le k\le r}\sum_{1\le j\le m_{13}}\Bigg(m_2\beta B\frac{1}{4}\left(\gamma^k B+\frac{\gamma^k}{\beta}-1\right)\left(A\varepsilon^j+\frac{\varepsilon^j}{a}-1\right)\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)-\left(\frac{(r+2)\alpha-\Gamma}{2\alpha}-\frac{Z-m_{12}\gamma}{\gamma}-\frac{\Delta-m_{23}\alpha}{\alpha}+\frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\varepsilon^j}{\alpha}-1\right)\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)\Bigg)$$

$$\ge \sum_{1\le k\le r}\sum_{1\le j\le m_{13}}\Bigg(m_2\beta B\frac{1}{4}\left(\frac{\gamma^k}{\beta}-1\right)\left(\frac{\varepsilon^j}{a}-1\right)\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)-\left(\frac{(r+2)\alpha-\Gamma}{2\alpha}-\frac{Z-m_{12}\gamma}{\gamma}-\frac{\Delta-m_{23}\alpha}{\alpha}+\frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\left(\frac{\gamma^k}{\beta}-1\right)\left(\frac{\varepsilon^j}{\alpha}-1\right)\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)\Bigg) \quad \left(\text{Αφού } \frac{\gamma^k}{\beta}-1\ge\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)$$

$$= \sum_{1\le k\le r}\sum_{1\le j\le m_{13}}\Bigg(\left(\frac{\gamma^k}{\beta}-1\right)\left(\frac{\varepsilon^j}{a}-1\right)\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)\Bigg(\frac{m_2\beta B}{4}-\left(\frac{(r+2)\alpha-\Gamma}{2\alpha}-\frac{Z-m_{12}\gamma}{\gamma}-\frac{\Delta-m_{23}\alpha}{\alpha}+\frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\Bigg)\Bigg)$$

$$= \left(\frac{m_2\beta B}{4}-\left(\frac{(r+2)\alpha-\Gamma}{2\alpha}-\frac{Z-m_{12}\gamma}{\gamma}-\frac{\Delta-m_{23}\alpha}{\alpha}+\frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\right)\sum_{1\le k\le r}\sum_{1\le j\le m_{13}}\left(\left(\frac{\gamma^k}{\beta}-1\right)\left(\frac{\varepsilon^j}{a}-1\right)\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)\right)$$

$$= \left(\frac{m_2\beta B}{4}-\left(\frac{(r+2)\alpha-\Gamma}{2\alpha}-\frac{Z-m_{12}\gamma}{\gamma}-\frac{\Delta-m_{23}\alpha}{\alpha}+\frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\right)\sum_{1\le k\le r}\sum_{1\le j\le m_{13}}\left(\left(\frac{\gamma^k-\beta}{\beta}\right)\left(\frac{\gamma^k-\beta}{\gamma^k\beta}\right)\left(\frac{\varepsilon^j-a}{a}\right)\right)$$

$$= \left(\frac{m_2\beta B}{4}-\left(\frac{(r+2)\alpha-\Gamma}{2\alpha}-\frac{Z-m_{12}\gamma}{\gamma}-\frac{\Delta-m_{23}\alpha}{\alpha}+\frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\right)\frac{1}{\alpha\beta^2}\sum_{1\le k\le r}\sum_{1\le j\le m_{13}}\left(\frac{\left(\gamma^k-\beta\right)^2}{\gamma^k}\left(\varepsilon^j-a\right)\right)$$

$$\geq \left(\frac{m_2\beta B}{4} - \left(\frac{(r+2)\alpha-\Gamma}{2\alpha} - \frac{Z-m_{12}\gamma}{\gamma} - \frac{\Delta-m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\right)\frac{1}{\alpha\beta^2}\sum_{1\leq k\leq r}\sum_{1\leq j\leq m_{13}}\left(\frac{\left(\gamma^k-\gamma\right)^2}{\gamma^k}\left(\varepsilon^j - a\right)\right).$$

(Αφού $\gamma \geq \beta$)

7. <u>Απλοποίηση της Δ_7</u>

$$\Delta 7 = \sum_{1\leq i\leq m_2} f^{r+m_1+i} \sum_{1\leq \kappa<j\leq m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\beta} - \frac{1}{\varepsilon^k}\right)$$
$$- \sum_{1\leq i\leq m_2} p_2^{r+m_1+i} \sum_{1\leq \kappa<j\leq m_{13}} p_3^{r+m_1+m_2+m_3+m_{12}+k} p_1^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\beta} - \frac{1}{\varepsilon^k}\right)$$

$$= \sum_{1\leq i\leq m_2}\sum_{1\leq \kappa<j\leq m_{13}}\left(f^{r+m_1+i} f^{r+m_1+m_2+m_3+m_{12}+k} f^{r+m_1+m_2+m_3+m_{12}+j}\right.$$
$$\left.- p_2^{r+m_1+i} p_3^{r+m_1+m_2+m_3+m_{12}+k} p_1^{r+m_1+m_2+m_3+m_{12}+j}\right)\left(\frac{1}{\beta} - \frac{1}{\varepsilon^k}\right)$$

$$= \sum_{1\leq \kappa<j\leq m_{13}}\left(m_2 f^{r+m_1+1} f^{r+m_1+m_2+m_3+m_{12}+k} f^{r+m_1+m_2+m_3+m_{12}+j} - \left(\frac{(r+2)\alpha-\Gamma}{2\alpha} - \right.\right.$$
$$\left.\left.\frac{Z-m_{12}\gamma}{\gamma} - \frac{\Delta-m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma-\beta)}{2\beta}\right) p_3^{r+m_1+m_2+m_3+m_{12}+k} p_1^{r+m_1+m_2+m_3+m_{12}+j}\right)\left(\frac{1}{\beta} - \frac{1}{\varepsilon^k}\right)$$

$$= \sum_{1\leq \kappa<j\leq m_{13}}\left(m_2\beta B\frac{1}{2}\left(A\varepsilon^k + \frac{\varepsilon^k}{\gamma} - 1\right)\frac{1}{2}\left(A\varepsilon^j + \frac{\varepsilon^j}{a} - 1\right) - \left(\frac{(r+2)\alpha-\Gamma}{2\alpha} - \frac{Z-m_{12}\gamma}{\gamma} - \right.\right.$$
$$\left.\left.\frac{\Delta-m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\left(\frac{\varepsilon^j}{\alpha} - 1\right)\left(\frac{\varepsilon^k}{\gamma} - 1\right)\right)\left(\frac{1}{\beta} - \frac{1}{\varepsilon^k}\right)$$

$$= \sum_{1\leq \kappa<j\leq m_{13}}\left(m_2\beta B\frac{1}{4}\left(A\varepsilon^k + \frac{\varepsilon^k}{\gamma} - 1\right)\left(A\varepsilon^j + \frac{\varepsilon^j}{a} - 1\right)\left(\frac{1}{\beta} - \frac{1}{\varepsilon^k}\right) - \left(\frac{(r+2)\alpha-\Gamma}{2\alpha} - \right.\right.$$
$$\left.\left.\frac{Z-m_{12}\gamma}{\gamma} - \frac{\Delta-m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\left(\frac{\varepsilon^j}{\alpha} - 1\right)\left(\frac{\varepsilon^k}{\gamma} - 1\right)\left(\frac{1}{\beta} - \frac{1}{\varepsilon^k}\right)\right)$$

$$= \sum_{1\leq \kappa<j\leq m_{13}}\left(m_2\beta B\frac{1}{4}\left(A\varepsilon^k + \frac{\varepsilon^k}{\gamma} - 1\right)\left(A\varepsilon^j + \frac{\varepsilon^j}{a} - 1\right)\left(\frac{1}{\beta} - \frac{1}{\varepsilon^k}\right) - \left(\frac{(r+2)\alpha-\Gamma}{2\alpha} - \right.\right.$$
$$\left.\left.\frac{Z-m_{12}\gamma}{\gamma} - \frac{\Delta-m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\left(\frac{\varepsilon^j}{\alpha} - 1\right)\left(\frac{\varepsilon^k}{\gamma} - 1\right)\left(\frac{1}{\beta} - \frac{1}{\varepsilon^k}\right)\right)$$

$$\geq \sum_{1\leq\kappa<j\leq m_{13}}\left(\left(\frac{\varepsilon^k}{\gamma}-1\right)\left(\frac{\varepsilon^j}{a}-1\right)\left(\frac{1}{\beta}-\frac{1}{\varepsilon^k}\right)\left(\frac{m_2\beta B}{4}-\left(\frac{(r+2)\alpha-\Gamma}{2\alpha}-\frac{Z-m_{12}\gamma}{\gamma}-\frac{\Delta-m_{23}\alpha}{\alpha}+\frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\right)\right)$$

$$=\left(\frac{m_2\beta B}{4}-\left(\frac{(r+2)\alpha-\Gamma}{2\alpha}-\frac{Z-m_{12}\gamma}{\gamma}-\frac{\Delta-m_{23}\alpha}{\alpha}+\frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\right)\sum_{1\leq\kappa<j\leq m_{13}}\left(\left(\frac{\varepsilon^k-\gamma}{\gamma}\right)\left(\frac{\varepsilon^j-\alpha}{a}\right)\left(\frac{\varepsilon^k-\beta}{\varepsilon^k\beta}\right)\right)$$

$$=\left(\frac{m_2\beta B}{4}-\left(\frac{(r+2)\alpha-\Gamma}{2\alpha}-\frac{Z-m_{12}\gamma}{\gamma}-\frac{\Delta-m_{23}\alpha}{\alpha}+\frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\right)\frac{1}{\alpha\beta\gamma}\sum_{1\leq\kappa<j\leq m_{13}}\left((\varepsilon^k-\gamma)(\varepsilon^j-\alpha)\left(\frac{\varepsilon^k-\beta}{\varepsilon^k}\right)\right)$$

$$\geq\left(\frac{m_2\beta B}{4}-\left(\frac{(r+2)\alpha-\Gamma}{2\alpha}-\frac{Z-m_{12}\gamma}{\gamma}-\frac{\Delta-m_{23}\alpha}{\alpha}+\frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\right)\frac{1}{\alpha\beta\gamma}\sum_{1\leq\kappa<j\leq m_{13}}\left((\varepsilon^j-\alpha)\left(\frac{(\varepsilon^k-\gamma)^2}{\varepsilon^k}\right)\right)$$

(Αφού $\gamma\geq\beta$)

8. <u>Απλοποίηση της Δ_8</u>

$$\Delta 8=\sum_{1\leq k\leq r} f^k\sum_{1\leq i\leq m_2} f^{r+m_1+i}\sum_{1\leq j\leq r} f^j\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)-\sum_{1\leq k\leq r} p_3^k\sum_{1\leq i\leq m_2} p_2^{r+m_1+i}\sum_{1\leq j\leq r} p_1^j\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)$$

$$=\sum_{1\leq i\leq m_2} f^{r+m_1+i}\sum_{1\leq\kappa<j\leq r} f^kf^j\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)-\sum_{1\leq i\leq m_2} p_2^{r+m_1+i}\sum_{1\leq\kappa<j\leq r} p_3^kp_1^j\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)$$

$$=\sum_{1\leq i\leq m_2}\sum_{1\leq\kappa<j\leq r}\left(f^{r+m_1+i}f^kf^j-p_2^{r+m_1+i}p_3^kp_1^j\right)\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)$$

$$=\sum_{1\leq\kappa<j\leq r}\left(m_2\beta B\frac{1}{2}\left(\gamma^kB+\frac{\gamma^k}{\beta}-1\right)\frac{1}{2}\left(\gamma^jB+\frac{\gamma^j}{\beta}-1\right)-\left(\frac{(r+2)\alpha-\Gamma}{2\alpha}-\frac{Z-m_{12}\gamma}{\gamma}-\frac{\Delta-m_{23}\alpha}{\alpha}+\frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\right)\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)$$

$$\geq\sum_{1\leq\kappa<j\leq r}\left(\frac{m_2\beta B}{4}\left(\frac{\gamma^k}{\beta}-1\right)\left(\frac{\gamma^j}{\beta}-1\right)\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)-\left(\frac{(r+2)\alpha-\Gamma}{2\alpha}-\frac{Z-m_{12}\gamma}{\gamma}-\frac{\Delta-m_{23}\alpha}{\alpha}+\frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\left(\frac{\gamma^k}{\beta}-1\right)\left(\frac{\gamma^j}{\beta}-1\right)\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)\right)\quad\left(\text{Αφού }\frac{\gamma^k}{\beta}-1\geq\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)$$

$$=\sum_{1\leq\kappa<j\leq r}\left(\left(\frac{\gamma^k}{\beta}-1\right)\left(\frac{\gamma^j}{\beta}-1\right)\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)\left(\frac{m_2\beta B}{4}-\left(\frac{(r+2)\alpha-\Gamma}{2\alpha}-\frac{Z-m_{12}\gamma}{\gamma}-\frac{\Delta-m_{23}\alpha}{\alpha}+\frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\right)\right)$$

$$= \left(\frac{m_2\beta B}{4} - \left(\frac{(r+2)\alpha - \Gamma}{2\alpha} - \frac{Z - m_{12}\gamma}{\gamma} - \frac{\Delta - m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\right)\sum_{1\le\kappa<j\le r}\left(\left(\frac{\gamma^k}{\beta}-1\right)\left(\frac{\gamma^j}{\beta}-1\right)\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)\right)$$

$$= \left(\frac{m_2\beta B}{4} - \left(\frac{(r+2)\alpha - \Gamma}{2\alpha} - \frac{Z - m_{12}\gamma}{\gamma} - \frac{\Delta - m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\right)\sum_{1\le k<j\le r}\left(\left(\frac{\gamma^k-\beta}{\beta}\right)\left(\frac{\gamma^k-\beta}{\gamma^k\beta}\right)\left(\frac{\gamma^j-\beta}{\beta}\right)\right)$$

$$= \left(\frac{m_2\beta B}{4} - \left(\frac{(r+2)\alpha - \Gamma}{2\alpha} - \frac{Z - m_{12}\gamma}{\gamma} - \frac{\Delta - m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\right)\frac{1}{\beta^3}\sum_{1\le k<j\le r}\left(\frac{\left(\gamma^k-\beta\right)^2}{\gamma^k}\left(\gamma^j-\beta\right)\right).$$

$$\ge \left(\frac{m_2\beta B}{4} - \left(\frac{(r+2)\alpha - \Gamma}{2\alpha} - \frac{Z - m_{12}\gamma}{\gamma} - \frac{\Delta - m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\right)\frac{1}{\beta^3}\sum_{1\le k<j\le r}\left(\frac{\left(\gamma^k-\gamma\right)^2}{\gamma^k}\left(\gamma^j-\beta\right)\right).$$

(Αφού $\gamma \ge \beta$)

9. Απλοποίηση της Δ_9

$$\Delta 9 = \sum_{1\le k\le r} f^k \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j}\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^k}\right)$$

$$- \sum_{1\le k\le r} p_3^k \sum_{1\le i\le m_{12}} p_2^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_{12}} p_1^{r+m_1+m_2+m_3+j}\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^k}\right)$$

$$= \sum_{1\le k\le r}\sum_{1\le j<i\le m_{12}} f^k f^{r+m_1+m_2+m_3+i} f^{r+m_1+m_2+m_3+j}\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^k}\right)$$

$$- \sum_{1\le k\le r}\sum_{1\le j<i\le m_{12}} p_3^k p_2^{r+m_1+m_2+m_3+i} p_1^{r+m_1+m_2+m_3+j}\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^k}\right)$$

$$= \sum_{1\le k\le r}\sum_{1\le j<i\le m_{12}} \left(f^k f^{r+m_1+m_2+m_3+i} f^{r+m_1+m_2+m_3+j} - p_3^k p_2^{r+m_1+m_2+m_3+i} p_1^{r+m_1+m_2+m_3+j}\right)\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^k}\right)$$

$$= \sum_{1\le k\le r}\sum_{1\le j<i\le m_{12}} \left(\frac{1}{2}\left(\gamma^k B + \frac{\gamma^k}{\beta} - 1\right)\frac{1}{2}\left(G\zeta^i + \frac{\zeta^i}{\gamma} - 1\right)\frac{1}{2}\left(B\zeta^j + \frac{\zeta^j}{\beta} - 1\right)\right.$$
$$\left. - \left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2}\right)\left(\frac{\zeta^i}{\gamma} - 1\right)\left(\frac{\zeta^j}{\beta} - 1\right)\right)\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^k}\right)$$

$$= \sum_{1\le k\le r}\sum_{1\le j<i\le m_{12}} \left(\frac{1}{8}\left(\gamma^k B + \frac{\gamma^k}{\beta} - 1\right)\left(G\zeta^i + \frac{\zeta^i}{\gamma} - 1\right)\left(B\zeta^j + \frac{\zeta^j}{\beta} - 1\right)\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^k}\right)\right.$$
$$\left. - \left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2}\right)\left(\frac{\zeta^i}{\gamma} - 1\right)\left(\frac{\zeta^j}{\beta} - 1\right)\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^k}\right)\right)$$

Έχουμε ότι για $k \in [r], p_1^k = \frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2}$ όπου $0 < p_1^k < 1$. Έτσι $p_1^k + p_3^k = \frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2} + \frac{\gamma^k}{2\beta} + \frac{\gamma^k}{2\alpha} - \frac{\gamma^k}{2\gamma} - \frac{1}{2} = \frac{\gamma^k}{\beta} - 1$. Επομένως, $\frac{\gamma^k}{\beta} - 1 \geq p_3^k$. Άρα,

$$
\begin{aligned}
\Delta_9 \geq \sum_{1\leq k\leq r}\sum_{1\leq j<i\leq m_{12}} &\left(\frac{1}{8}\left(\gamma^k B + p_3^k\right)\left(G\zeta^i + p_2^i\right)\left(B\zeta^j + p_1^j\right)\left(\frac{1}{\zeta^i} - \frac{1}{\gamma^k}\right)\right. \\
&\left. - p_3^k p_2^i p_1^j \left(\frac{1}{\zeta^i} - \frac{1}{\gamma^k}\right)\right)
\end{aligned}
$$

$$
= \sum_{1\leq k\leq r}\sum_{1\leq j<i\leq m_{12}} \left(\frac{1}{8}\left(\gamma^k B + p_3^k\right)\left(G\zeta^i + p_2^i\right)\left(B\zeta^j + p_1^j\right) - p_3^k p_2^i p_1^j\right)\left(\frac{1}{\zeta^i} - \frac{1}{\gamma^k}\right)
$$

$$
\begin{aligned}
= \sum_{1\leq k\leq r}\sum_{1\leq j<i\leq m_{12}} &\left(\frac{1}{8}\left(\gamma^k BG\zeta^i + \gamma^k B p_2^i + G\zeta^i p_3^k + p_3^k p_2^i\right)\left(B\zeta^j + p_1^j\right) - p_3^k p_2^i p_1^j\right)\left(\frac{1}{\zeta^i}\right. \\
&\left. - \frac{1}{\gamma^k}\right)
\end{aligned}
$$

$$
\begin{aligned}
= \sum_{1\leq k\leq r}\sum_{1\leq j<i\leq m_{12}} &\left(\frac{1}{8}\left(\gamma^k BG\zeta^i B\zeta^j + \gamma^k BG\zeta^i p_1^j + \gamma^k B p_2^i B\zeta^j + \gamma^k B p_2^i p_1^j + G\zeta^i p_3^k B\zeta^j\right.\right. \\
&\left.\left. + G\zeta^i p_3^k p_1^j + B\zeta^j p_3^k p_2^i - 7 p_3^k p_2^i p_1^j\right)\right)\left(\frac{1}{\zeta^i} - \frac{1}{\gamma^k}\right)
\end{aligned}
$$

$$
\begin{aligned}
= \sum_{1\leq k\leq r}\sum_{1\leq j<i\leq m_{12}} &\left(\frac{1}{8}\left(\gamma^k B^2 G\zeta^i\zeta^j + \gamma^k BG\zeta^i\left(\frac{\zeta^j}{\beta} - 1\right) + \gamma^k B^2\zeta^j\left(\frac{\zeta^i}{\gamma} - 1\right)\right.\right. \\
&+ \gamma^k B\left(\frac{\zeta^i}{\gamma} - 1\right)\left(\frac{\zeta^j}{\beta} - 1\right) + G\zeta^i\left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2}\right)B\zeta^j \\
&+ G\zeta^i\left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2}\right)\left(\frac{\zeta^j}{\beta} - 1\right) + B\zeta^j\left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2}\right)\left(\frac{\zeta^i}{\gamma} - 1\right) \\
&\left.\left. - 7\left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2}\right)\left(\frac{\zeta^i}{\gamma} - 1\right)\left(\frac{\zeta^j}{\beta} - 1\right)\right)\right)\left(\frac{1}{\zeta^i} - \frac{1}{\gamma^k}\right)
\end{aligned}
$$

$$\geq \sum_{1\leq k\leq r}\sum_{1\leq j<i\leq m_{12}}\left(\frac{1}{8}\left(\gamma^k B^3\zeta^i\zeta^j+\gamma^k B^2\zeta^i\left(\frac{\zeta^j}{\beta}-1\right)+\gamma^k B^2\zeta^j\left(\frac{\zeta^i}{\gamma}-1\right)\right.\right.$$
$$+\gamma^k B\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{\zeta^j}{\beta}-1\right)+B^2\zeta^i\zeta^j\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)$$
$$+B\zeta^i\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\zeta^j}{\beta}-1\right)+B\zeta^j\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\zeta^i}{\gamma}-1\right)$$
$$\left.\left.-7\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{\zeta^j}{\beta}-1\right)\right)\right)\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^k}\right)\ (\text{Αφού } G\geq B)$$

$$=\frac{1}{8}\sum_{1\leq k\leq r}\sum_{1\leq j<i\leq m_{12}}\left(B^3\zeta^j\gamma^k\left(1-\frac{\zeta^i}{\gamma^k}\right)+B^2\gamma^k\left(\frac{\zeta^j}{\beta}-1\right)\left(1-\frac{\zeta^i}{\gamma^k}\right)\right.$$
$$+B^2\zeta^j\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{\gamma^k}{\zeta^i}-1\right)+B\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{\zeta^j}{\beta}-1\right)\left(\frac{\gamma^k}{\zeta^i}-1\right)$$
$$+B^2\zeta^j\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(1-\frac{\zeta^i}{\gamma^k}\right)$$
$$+B\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\zeta^j}{\beta}-1\right)\left(1-\frac{\zeta^i}{\gamma^k}\right)$$
$$+B\frac{\zeta^j}{\gamma^k}\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{\gamma^k}{\zeta^i}-1\right)$$
$$\left.-\frac{7}{\gamma^k}\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{\zeta^j}{\beta}-1\right)\left(\frac{\gamma^k}{\zeta^i}-1\right)\right)$$

$$=\frac{1}{8}\sum_{1\leq k\leq r}\sum_{1\leq j\leq m_{12}}\left(\left(B^3\zeta^j\gamma^k+B^2\gamma^k\left(\frac{\zeta^j}{\beta}-1\right)+B^2\zeta^j\left(\frac{\zeta^j}{\gamma}-1\right)+B\left(\frac{\zeta^j}{\gamma}-1\right)\left(\frac{\zeta^j}{\beta}-1\right)\right.\right.$$
$$+B^2\zeta^j\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)+B\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\zeta^j}{\beta}-1\right)$$
$$+B\frac{\zeta^j}{\gamma^k}\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\zeta^j}{\gamma}-1\right)$$
$$\left.\left.-\frac{7}{\gamma^k}\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\zeta^j}{\gamma}-1\right)\left(\frac{\zeta^j}{\beta}-1\right)\right)\sum_{1\leq i\leq m_{12}}\left|1-\frac{\zeta^i}{\gamma^k}\right|\right)$$

$$\geq \frac{1}{8} \sum_{1 \leq k \leq r} \sum_{1 \leq j \leq m_{12}} \left(\left(B^3 \zeta^j + B^2 \gamma^k \left(\frac{\zeta^j}{\gamma} - 1 \right) + B^2 \zeta^j \left(\frac{\zeta^j}{\gamma} - 1 \right) + B \left(\frac{\zeta^j}{\gamma} - 1 \right) \left(\frac{\zeta^j}{\gamma} - 1 \right) \right. \right.$$

$$+ B^2 \zeta^j \left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2} \right) + B \left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2} \right) \left(\frac{\zeta^j}{\gamma} - 1 \right)$$

$$+ B \frac{\zeta^j}{\gamma^k} \left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2} \right) \left(\frac{\zeta^j}{\gamma} - 1 \right)$$

$$\left. \left. - \frac{7}{\gamma^k} \left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2} \right) \left(\frac{\zeta^j}{\gamma} - 1 \right) \left(\frac{\zeta^j}{\gamma} - 1 \right) \right) \frac{1}{\gamma^k} \sum_{1 \leq i \leq m_{12}} \left| \gamma^k - \zeta^i \right| \right) (\text{Αφού } \gamma \geq \beta)$$

$$= \frac{1}{8} \sum_{1 \leq k \leq r} \sum_{1 \leq j \leq m_{12}} \left(\left(B^3 \zeta^j + B^2 \gamma^k \left(\frac{\zeta^j}{\gamma} - 1 \right) + B^2 \zeta^j \left(\frac{\zeta^j}{\gamma} - 1 \right) + B \left(\frac{\zeta^j}{\gamma} - 1 \right) \left(\frac{\zeta^j}{\gamma} - 1 \right) \right. \right.$$

$$+ B^2 \zeta^j \left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2} \right) + B \left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2} \right) \left(\frac{\zeta^j}{\gamma} - 1 \right)$$

$$+ \frac{B \zeta^j}{\gamma^k} \left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2} \right) \left(\frac{\zeta^j}{\gamma} - 1 \right)$$

$$\left. \left. - \frac{7}{\gamma^k} \left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2} \right) \left(\frac{\zeta^j}{\gamma} - 1 \right) \left(\frac{\zeta^j}{\gamma} - 1 \right) \right) \frac{1}{\gamma^k} S(k) \right)$$

$$= \frac{1}{8} \sum_{1 \leq k \leq r} \sum_{1 \leq j \leq m_{12}} \left(B^3 \zeta^j + B \left(\frac{\zeta^j}{\gamma} - 1 \right) \left(\gamma^k B + B \zeta^j + \left(\frac{\zeta^j}{\gamma} - 1 \right) \right) \right)$$

$$+ \left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2} \right) \left(B^2 \zeta^j + B \left(\frac{\zeta^j}{\gamma} - 1 \right) + B \zeta^j \left(\frac{\zeta^j}{\gamma} - 1 \right) \right.$$

$$\left. \left. - 7 \left(\frac{\zeta^j}{\gamma} - 1 \right) \left(\frac{\zeta^j}{\gamma} - 1 \right) \right) \frac{1}{\gamma^k} S(k) \right)$$

10. <u>Απλοποίηση της Δ_{10}</u>

$$\Delta 10 = \sum_{1 \leq k \leq r} f^k \sum_{1 \leq i \leq m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1 \leq j \leq r} f^j \left(\frac{1}{\zeta^i} - \frac{1}{\gamma^k} \right)$$

$$- \sum_{1 \leq k \leq r} p_3^k \sum_{1 \leq i \leq m_{12}} p_2^{r+m_1+m_2+m_3+i} \sum_{1 \leq j \leq r} p_1^j \left(\frac{1}{\zeta^i} - \frac{1}{\gamma^k} \right)$$

$$= \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\le k<j\le r} f^k f^j \left(\frac{1}{\zeta^i} - \frac{1}{\gamma^k}\right)$$
$$- \sum_{1\le i\le m_{12}} p_2^{r+m_1+m_2+m_3+i} \sum_{1\le k<j\le r} p_3^k p_1^j \left(\frac{1}{\zeta^i} - \frac{1}{\gamma^k}\right)$$

$$= \sum_{1\le i\le m_{12}} \sum_{1\le k<j\le r} \left(f^{r+m_1+m_2+m_3+i} f^k f^j - p_2^{r+m_1+m_2+m_3+i} p_3^k p_1^j\right) \left(\frac{1}{\zeta^i} - \frac{1}{\gamma^k}\right)$$
$$= \sum_{1\le i\le m_{12}} \sum_{1\le k<j\le r} \left(\frac{1}{2}\left(G\zeta^i + p_2^i\right) \frac{1}{2}\left(\gamma^k B + \frac{\gamma^k}{\beta} - 1\right) \frac{1}{2}\left(\gamma^j B + \frac{\gamma^j}{\beta} - 1\right) - p_2^i p_3^k p_1^j\right) \left(\frac{1}{\zeta^i} - \frac{1}{\gamma^k}\right)$$
$\geq \sum_{1\le i\le m_{12}} \sum_{1\le k<j\le r} \left(\frac{1}{2}\left(G\zeta^i + p_2^i\right) \frac{1}{2}\left(\gamma^k B + p_3^k\right) \frac{1}{2}\left(\gamma^j B + p_1^j\right) - p_2^i p_3^k p_1^j\right) \left(\frac{1}{\zeta^i} - \frac{1}{\gamma^k}\right)$(Αφού $\frac{\gamma^k}{\beta} - 1 \geq p_3^k$ και $\frac{\gamma^j}{\beta} - 1 \geq p_1^j$)

$$= \sum_{1\le i\le m_{12}} \sum_{1\le k<j\le r} \left(\frac{1}{8}\left(G\zeta^i + p_2^i\right)\left(\gamma^k B + p_3^k\right)\left(\gamma^j B + p_1^j\right) - p_2^i p_3^k p_1^j\right) \left(\frac{1}{\zeta^i} - \frac{1}{\gamma^k}\right)$$
$$= \sum_{1\le i\le m_{12}} \sum_{1\le k<j\le r} \left(\frac{1}{8}\left(G\zeta^i\gamma^k B + G\zeta^i p_3^k + \gamma^k B p_2^i + p_2^i p_3^k\right)\left(\gamma^j B + p_1^j\right) - p_2^i p_3^k p_1^j\right) \left(\frac{1}{\zeta^i} - \frac{1}{\gamma^k}\right)$$
$$= \sum_{1\le i\le m_{12}} \sum_{1\le k<j\le r} \Bigg(\frac{1}{8}\Big(G\zeta^i\gamma^k B\gamma^j B + G\zeta^i\gamma^k B p_1^j + G\zeta^i p_3^k \gamma^j B + G\zeta^i p_3^k p_1^j + \gamma^k B p_2^i \gamma^j B$$
$$+ \gamma^k B p_2^i p_1^j + \gamma^j B p_2^i p_3^k - 7 p_2^i p_3^k p_1^j\Big)\Bigg) \left(\frac{1}{\zeta^i} - \frac{1}{\gamma^k}\right)$$

$$= \sum_{1 \le i \le m_{12}} \sum_{1 \le k < j \le r} \left(\frac{1}{8} \left(G\zeta^i \gamma^k B \gamma^j B + G\zeta^i \gamma^k B \left(\frac{\gamma^j}{2\beta} + \frac{\gamma^j}{2\alpha} - \frac{\gamma^j}{2\gamma} - \frac{1}{2} \right) \right.\right.$$

$$+ G\zeta^i \left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2} \right) \gamma^j B$$

$$+ G\zeta^i \left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2} \right) \left(\frac{\gamma^j}{2\beta} + \frac{\gamma^j}{2\alpha} - \frac{\gamma^j}{2\gamma} - \frac{1}{2} \right) + \gamma^k B \left(\frac{\zeta^i}{\gamma} - 1 \right) \gamma^j B$$

$$+ \gamma^k B \left(\frac{\zeta^i}{\gamma} - 1 \right) \left(\frac{\gamma^j}{2\beta} + \frac{\gamma^j}{2\alpha} - \frac{\gamma^j}{2\gamma} - \frac{1}{2} \right) + \gamma^j B \left(\frac{\zeta^i}{\gamma} - 1 \right) \left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2} \right)$$

$$\left.\left. - 7 \left(\frac{\zeta^i}{\gamma} - 1 \right) \left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2} \right) \left(\frac{\gamma^j}{2\beta} + \frac{\gamma^j}{2\alpha} - \frac{\gamma^j}{2\gamma} - \frac{1}{2} \right) \right) \right) \left(\frac{1}{\zeta^i} - \frac{1}{\gamma^k} \right)$$

$$= \sum_{1 \le i \le m_{12}} \sum_{1 \le k < j \le r} \left(\frac{1}{8} \left(G\zeta^i \gamma^k B \gamma^j B \left(\frac{1}{\zeta^i} - \frac{1}{\gamma^k} \right) + G\zeta^i \gamma^k B \left(\frac{\gamma^j}{2\beta} + \frac{\gamma^j}{2\alpha} - \frac{\gamma^j}{2\gamma} - \frac{1}{2} \right) \left(\frac{1}{\zeta^i} - \frac{1}{\gamma^k} \right) \right.\right.$$

$$+ G\zeta^i \left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2} \right) \gamma^j B \left(\frac{1}{\zeta^i} - \frac{1}{\gamma^k} \right)$$

$$+ G\zeta^i \left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2} \right) \left(\frac{\gamma^j}{2\beta} + \frac{\gamma^j}{2\alpha} - \frac{\gamma^j}{2\gamma} - \frac{1}{2} \right) \left(\frac{1}{\zeta^i} - \frac{1}{\gamma^k} \right)$$

$$+ \gamma^k B \left(\frac{\zeta^i}{\gamma} - 1 \right) \gamma^j B \left(\frac{1}{\zeta^i} - \frac{1}{\gamma^k} \right)$$

$$+ \gamma^k B \left(\frac{\zeta^i}{\gamma} - 1 \right) \left(\frac{\gamma^j}{2\beta} + \frac{\gamma^j}{2\alpha} - \frac{\gamma^j}{2\gamma} - \frac{1}{2} \right) \left(\frac{1}{\zeta^i} - \frac{1}{\gamma^k} \right)$$

$$+ \gamma^j B \left(\frac{\zeta^i}{\gamma} - 1 \right) \left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2} \right) \left(\frac{1}{\zeta^i} - \frac{1}{\gamma^k} \right)$$

$$\left.\left. - 7 \left(\frac{\zeta^i}{\gamma} - 1 \right) \left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2} \right) \left(\frac{\gamma^j}{2\beta} + \frac{\gamma^j}{2\alpha} - \frac{\gamma^j}{2\gamma} - \frac{1}{2} \right) \left(\frac{1}{\zeta^i} - \frac{1}{\gamma^k} \right) \right) \right)$$

$$
\begin{aligned}
\geq \sum_{1\leq i\leq m_{12}} \sum_{1\leq k<j\leq r} \Bigg(\frac{1}{8}\Bigg(& B^3\gamma^k\gamma^j\left(1-\frac{\zeta^i}{\gamma^k}\right) + B^2\gamma^k\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(1-\frac{\zeta^i}{\gamma^k}\right) \\
& + B^2\gamma^j\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(1-\frac{\zeta^i}{\gamma^k}\right) \\
& + B\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(1-\frac{\zeta^i}{\gamma^k}\right) + B^2\gamma^j\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{\gamma^k}{\zeta^i}-1\right) \\
& + B\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(\frac{\gamma^k}{\zeta^i}-1\right) \\
& + \frac{B\gamma^j}{\gamma^k}\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\gamma^k}{\zeta^i}-1\right) \\
& - \frac{7}{\gamma^k}\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(\frac{\gamma^k}{\zeta^i}-1\right)\Bigg)\Bigg) (\textit{Αφού } G \geq B)
\end{aligned}
$$

$$
\begin{aligned}
= \frac{1}{8}\sum_{1\leq i\leq m_{12}} \sum_{1\leq j\leq r} \Bigg(\Bigg(& B^3\gamma^j\gamma^j + B^2\gamma^j\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right) + B^2\gamma^j\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right) \\
& + B\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right) + B^2\gamma^j\left(\frac{\zeta^i}{\gamma}-1\right) \\
& + B\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right) + \frac{B\gamma^j}{\gamma^j}\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right) \\
& - \frac{7}{\gamma^j}\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\Bigg)\sum_{1\leq k\leq r}\left|\frac{\gamma^k}{\zeta^i}-1\right|\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
\geq \frac{1}{8}\sum_{1\leq i\leq m_{12}} \sum_{1\leq j\leq r} \Bigg(\Bigg(& B^3\gamma^j\gamma^j + B^2\gamma^j\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right) + B^2\gamma^j\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right) \\
& + B\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right) + B^2\gamma^j\left(\frac{\zeta^i}{\gamma}-1\right) \\
& + B\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right) + \frac{B\gamma^j}{\gamma^j}\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right) \\
& - \frac{7}{\gamma^j}\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma} \right. \\
& \left. -\frac{1}{2}\right)\Bigg)\frac{1}{\zeta^i}S(i)\Bigg)\left(\textit{Αφού } \left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right) \geq \left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\right)
\end{aligned}
$$

$$= \frac{1}{8} \sum_{1 \le i \le m_{12}} \sum_{1 \le j \le r} \left(\left(B^3 \gamma^j \gamma^j \right. \right.$$

$$+ \left(\frac{\gamma^j}{2\beta} - \frac{\gamma^j}{2\alpha} + \frac{\gamma^j}{2\gamma} - \frac{1}{2} \right) \left(2B^2\gamma^j + B\left(\frac{\gamma^j}{2\beta} - \frac{\gamma^j}{2\alpha} + \frac{\gamma^j}{2\gamma} - \frac{1}{2} \right) + B^2\gamma^j \left(\frac{\zeta^i}{\gamma} + 1 \right) + B\left(\frac{\zeta^i}{\gamma} - 1 \right) \right.$$

$$\left. \left. \left. + \frac{B\gamma^j}{\gamma^j} \left(\frac{\zeta^i}{\gamma} - 1 \right) - \frac{7}{\gamma^j} \left(\frac{\zeta^i}{\gamma} - 1 \right) \left(\frac{\gamma^j}{2\beta} - \frac{\gamma^j}{2\alpha} + \frac{\gamma^j}{2\gamma} - \frac{1}{2} \right) \right) \right) \frac{1}{\zeta^i} S(i) \right)$$

$$\ge \frac{1}{8} \sum_{1 \le i \le m_{12}} \sum_{1 \le j \le r} \left(\left(B^3 \gamma^j \gamma^j \right. \right.$$

$$+ \left(\frac{\gamma^j}{2\beta} - \frac{\gamma^j}{2\alpha} + \frac{\gamma^j}{2\gamma} - \frac{1}{2} \right) \left(2B^2\gamma^j + 2B\left(\frac{\zeta^i}{\gamma} - 1 \right) + B\left(\frac{\gamma^j}{2\beta} - \frac{\gamma^j}{2\alpha} + \frac{\gamma^j}{2\gamma} - \frac{1}{2} \right) \right.$$

$$\left. \left. \left. - \frac{7}{\gamma^j} \left(\frac{\zeta^i}{\gamma} - 1 \right) \left(\frac{\gamma^j}{2\beta} - \frac{\gamma^j}{2\alpha} + \frac{\gamma^j}{2\gamma} - \frac{1}{2} \right) \right) \right) \frac{1}{\zeta^i} S(i) \right) \left(A\varphi o\acute{\upsilon} \left(\frac{\zeta^i}{\gamma} + 1 \right) \ge 2 \ \alpha\varphi o\acute{\upsilon} \ \frac{\zeta^i}{\gamma} \ge 1 \right)$$

11. Απλοποίηση της Δ_{11}

$$\Delta 11 = \sum_{1 \le k \le r} f^k \sum_{1 \le i \le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1 \le j \le m_{12}} f^{r+m_1+m_2+m_3+j} \left(\frac{1}{\delta^i} - \frac{1}{\gamma^k} \right)$$

$$- \sum_{1 \le k \le r} p_3^k \sum_{1 \le i \le m_{23}} p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1 \le j \le m_{12}} p_1^j \left(\frac{1}{\delta^i} - \frac{1}{\gamma^k} \right)$$

$$= \sum_{1 \le k \le r} \sum_{1 \le i \le m_{23}} \sum_{1 \le j \le m_{12}} \left(f^k f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} f^{r+m_1+m_2+m_3+j} \right.$$

$$\left. - p_3^k p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} p_1^{r+m_1+m_2+m_3+j} \right) \left(\frac{1}{\delta^i} - \frac{1}{\gamma^k} \right)$$

$$\ge \sum_{1 \le k \le r} \sum_{1 \le i \le m_{23}} \sum_{1 \le j \le m_{12}} \left(\frac{1}{2} \left(\gamma^k B + p_3^k \right) \frac{1}{2} \left(B\delta^i + p_2^i \right) \frac{1}{2} \left(B\zeta^j + p_1^j \right) - p_3^k p_2^i p_1^j \right) \left(\frac{1}{\delta^i} - \frac{1}{\gamma^k} \right)$$

$$= \sum_{1\le k\le r}\sum_{1\le i\le m_{23}}\sum_{1\le j\le m_{12}} \left(\frac{1}{8}\left(\gamma^k BB\delta^i + \gamma^k Bp_2^i + B\delta^i p_3^k + p_3^k p_2^i\right)\left(B\zeta^j + p_1^j\right) - p_3^k p_2^i p_1^j\right)\left(\frac{1}{\delta^i} - \frac{1}{\gamma^k}\right)$$

$$= \frac{1}{8}\sum_{1\le k\le r}\sum_{1\le i\le m_{23}}\sum_{1\le j\le m_{12}} \left(B^3\delta^i\zeta^j\gamma^k + \gamma^k B^2\delta^i p_1^j + B^2 p_2^i\zeta^j\gamma^k + B\gamma^k p_2^i p_1^j + B^2\delta^i p_3^k\zeta^j + B\delta^i p_3^k p_1^j \right.$$
$$\left. + p_3^k p_2^i B\zeta^j - 7p_3^k p_2^i p_1^j\right)\left(\frac{1}{\delta^i} - \frac{1}{\gamma^k}\right)$$

$$= \frac{1}{8}\sum_{1\le k\le r}\sum_{1\le i\le m_{23}}\sum_{1\le j\le m_{12}} \left(B^3\delta^i\zeta^j\gamma^k + B^2\gamma^k\delta^i\left(\frac{\zeta^j}{\beta} - 1\right) + B^2\zeta^j\gamma^k\left(\frac{\delta^i}{\alpha} - 1\right) + B\gamma^k\left(\frac{\delta^i}{\alpha} - 1\right)\left(\frac{\zeta^j}{\beta} - 1\right)\right.$$
$$+ B^2\delta^i\zeta^j\left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2}\right) + B\delta^i\left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2}\right)\left(\frac{\zeta^j}{\beta} - 1\right)$$
$$\left. + B\zeta^j\left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2}\right)\left(\frac{\delta^i}{\alpha} - 1\right) - 7\left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2}\right)\left(\frac{\delta^i}{\alpha} - 1\right)\left(\frac{\zeta^j}{\beta} - 1\right)\right)\left(\frac{1}{\delta^i}\right.$$
$$\left. - \frac{1}{\gamma^k}\right)$$

$$= \frac{1}{8}\sum_{1\le k\le r}\sum_{1\le i\le m_{23}}\sum_{1\le j\le m_{12}} \left(B^3\delta^i\zeta^j\gamma^k\left(\frac{1}{\delta^i} - \frac{1}{\gamma^k}\right) + B^2\gamma^k\delta^i\left(\frac{\zeta^j}{\beta} - 1\right)\left(\frac{1}{\delta^i} - \frac{1}{\gamma^k}\right)\right.$$
$$+ B^2\zeta^j\gamma^k\left(\frac{\delta^i}{\alpha} - 1\right)\left(\frac{1}{\delta^i} - \frac{1}{\gamma^k}\right) + B\gamma^k\left(\frac{\delta^i}{\alpha} - 1\right)\left(\frac{\zeta^j}{\beta} - 1\right)\left(\frac{1}{\delta^i} - \frac{1}{\gamma^k}\right)$$
$$+ B^2\delta^i\zeta^j\left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2}\right)\left(\frac{1}{\delta^i} - \frac{1}{\gamma^k}\right)$$
$$+ B\delta^i\left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2}\right)\left(\frac{\zeta^j}{\beta} - 1\right)\left(\frac{1}{\delta^i} - \frac{1}{\gamma^k}\right)$$
$$+ B\zeta^j\left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2}\right)\left(\frac{\delta^i}{\alpha} - 1\right)\left(\frac{1}{\delta^i} - \frac{1}{\gamma^k}\right)$$
$$\left. - 7\left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2}\right)\left(\frac{\delta^i}{\alpha} - 1\right)\left(\frac{\zeta^j}{\beta} - 1\right)\left(\frac{1}{\delta^i} - \frac{1}{\gamma^k}\right)\right)$$

$$
\begin{aligned}
=\frac{1}{8}\sum_{1\le k\le r}\sum_{1\le i\le m_{23}}\sum_{1\le j\le m_{12}} & \Bigg(B^3\zeta^j\gamma^k\left(1-\frac{\delta^i}{\gamma^k}\right)+B^2\gamma^k\left(\frac{\zeta^j}{\beta}-1\right)\left(1-\frac{\delta^i}{\gamma^k}\right)+B^2\zeta^j\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\gamma^k}{\delta^i}-1\right) \\
& +B\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\zeta^j}{\beta}-1\right)\left(\frac{\gamma^k}{\delta^i}-1\right)+B^2\zeta^j\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(1-\frac{\delta^i}{\gamma^k}\right) \\
& +B\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\zeta^j}{\beta}-1\right)\left(1-\frac{\delta^i}{\gamma^k}\right) \\
& +\frac{B\zeta^j}{\gamma^k}\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\gamma^k}{\delta^i}-1\right) \\
& -\frac{7}{\gamma^k}\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\zeta^j}{\beta}-1\right)\left(\frac{\gamma^k}{\delta^i}-1\right)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
\geq\frac{1}{8}\sum_{1\le k\le r}\sum_{1\le i\le m_{23}}\sum_{1\le j\le m_{12}} & \Bigg(B^3\gamma^k\left(\frac{\zeta^j}{\beta}-1\right)\left(1-\frac{\delta^i}{\gamma^k}\right)+B^2\gamma^k\left(\frac{\zeta^j}{\beta}-1\right)\left(1-\frac{\delta^i}{\gamma^k}\right) \\
& +B^2\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\zeta^j}{\beta}-1\right)\left(\frac{\gamma^k}{\delta^i}-1\right)+B\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\zeta^j}{\beta}-1\right)\left(\frac{\gamma^k}{\delta^i}-1\right) \\
& +B^2\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\zeta^j}{\beta}-1\right)\left(1-\frac{\delta^i}{\gamma^k}\right) \\
& +B\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\zeta^j}{\beta}-1\right)\left(1-\frac{\delta^i}{\gamma^k}\right) \\
& +\frac{B}{\gamma^k}\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\zeta^j}{\beta}-1\right)\left(\frac{\gamma^k}{\delta^i}-1\right) \\
& -\frac{7}{\gamma^k}\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\zeta^j}{\beta}-1\right)\left(\frac{\gamma^k}{\delta^i}-1\right)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
=\frac{1}{8}\sum_{1\le k\le r}\sum_{1\le i\le m_{23}}\sum_{1\le j\le m_{12}} & \Bigg(B^2\gamma^k\left(\frac{\zeta^j}{\beta}-1\right)\left(1-\frac{\delta^i}{\gamma^k}\right)(B+1)+B\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\zeta^j}{\beta}-1\right)\left(\frac{\gamma^k}{\delta^i}-1\right)(B+1) \\
& +B\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\zeta^j}{\beta}-1\right)\left(1-\frac{\delta^i}{\gamma^k}\right)(B+1) \\
& +\frac{1}{\gamma^k}\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\zeta^j}{\beta}-1\right)\left(\frac{\gamma^k}{\delta^i}-1\right)(B-7)\Bigg)
\end{aligned}
$$

$$= \frac{1}{8} \sum_{1\le k\le r} \sum_{1\le i\le m_{23}} \sum_{1\le j\le m_{12}} \left(B^2 \left(\frac{\zeta^j}{\beta} - 1\right)\left(\gamma^k - \delta^i\right)(B+1) \right.$$

$$+ \frac{B}{\gamma^k}\left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2}\right)\left(\frac{\zeta^j}{\beta} - 1\right)\left(\gamma^k - \delta^i\right)(B+1)$$

$$+ \frac{B}{\delta^i}\left(\frac{\delta^i}{\alpha} - 1\right)\left(\frac{\zeta^j}{\beta} - 1\right)\left(\gamma^k - \delta^i\right)(B+1)$$

$$\left. + \frac{1}{\gamma^k\delta^i}\left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2}\right)\left(\frac{\delta^i}{\alpha} - 1\right)\left(\frac{\zeta^j}{\beta} - 1\right)\left(\gamma^k - \delta^i\right)(B-7)\right)$$

12. Απλοποίηση της $\varDelta_{12}$

$$\varDelta 12 = \sum_{1\le k\le r} f^k \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\delta^i} - \frac{1}{\gamma^k}\right)$$

$$- \sum_{1\le k\le r} p_3^k \sum_{1\le i\le m_{23}} p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_{13}} p_1^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\delta^i} - \frac{1}{\gamma^k}\right)$$

$$= \sum_{1\le k\le r} \sum_{1\le i\le m_{23}} \sum_{1\le j\le m_{13}} \left(f^k f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} f^{r+m_1+m_2+m_3+m_{12}+j}\right.$$

$$\left. - p_3^k p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} p_1^{r+m_1+m_2+m_3+m_{12}+j}\right)\left(\frac{1}{\delta^i} - \frac{1}{\gamma^k}\right)$$

$$\ge \sum_{1\le k\le r} \sum_{1\le i\le m_{23}} \sum_{1\le j\le m_{13}} \left(\frac{1}{2}\left(\gamma^k B + p_3^k\right)\frac{1}{2}\left(B\delta^i + p_2^i\right)\frac{1}{2}\left(A\varepsilon^j + p_1^j\right) - p_3^k p_2^i p_1^j\right)\left(\frac{1}{\delta^i} - \frac{1}{\gamma^k}\right)$$

$$= \sum_{1\le k\le r} \sum_{1\le i\le m_{23}} \sum_{1\le j\le m_{13}} \left(\frac{1}{8}\left(\gamma^k BB\delta^i + \gamma^k B p_2^i + p_3^k B\delta^i + p_3^k p_2^i\right)\left(A\varepsilon^j + p_1^j\right) - p_3^k p_2^i p_1^j\right)\left(\frac{1}{\delta^i} - \frac{1}{\gamma^k}\right)$$

$$= \sum_{1\le k\le r} \sum_{1\le i\le m_{23}} \sum_{1\le j\le m_{13}} \left(\frac{1}{8}\left(\gamma^k BB\delta^i A\varepsilon^j + \gamma^k BB\delta^i p_1^j + \gamma^k B p_2^i A\varepsilon^j + \gamma^k B p_2^i p_1^j + p_3^k B\delta^i A\varepsilon^j\right.\right.$$

$$\left.\left. + p_3^k B\delta^i p_1^j + p_3^k p_2^i A\varepsilon^j + p_3^k p_2^i p_1^j\right) - p_3^k p_2^i p_1^j\right)\left(\frac{1}{\delta^i} - \frac{1}{\gamma^k}\right)$$

$$= \sum_{1\le k\le r} \sum_{1\le i\le m_{23}} \sum_{1\le j\le m_{13}} \left(\frac{1}{8}\left(AB^2\gamma^k\delta^i\varepsilon^j + B^2\gamma^k\delta^i p_1^j + AB\gamma^k p_2^i\varepsilon^j + B\gamma^k p_2^i p_1^j\right.\right.$$

$$\left.\left. + ABp_3^k\delta^i\varepsilon^j + Bp_3^k\delta^i p_1^j + Ap_3^k p_2^i\varepsilon^j + p_3^k p_2^i p_1^j\right) - p_3^k p_2^i p_1^j\right)\left(\frac{1}{\delta^i} - \frac{1}{\gamma^k}\right)$$

$$= \sum_{1\le k\le r}\sum_{1\le i\le m_{23}}\sum_{1\le j\le m_{13}}\Bigg(\frac{1}{8}\big(AB^2\gamma^k\delta^i\varepsilon^j + B^2\gamma^k\delta^i p_1^j + AB\gamma^k p_2^i\varepsilon^j + B\gamma^k p_2^i p_1^j$$
$$+ ABp_3^k\delta^i\varepsilon^j + Bp_3^k\delta^i p_1^j + Ap_3^k p_2^i\varepsilon^j + p_3^k p_2^i p_1^j\big) - p_3^k p_2^i p_1^j\Bigg)\left(\frac{1}{\delta^i} - \frac{1}{\gamma^k}\right)$$

$$= \frac{1}{8}\sum_{1\le k\le r}\sum_{1\le i\le m_{23}}\sum_{1\le j\le m_{13}}\Big(\big(AB^2\gamma^k\delta^i\varepsilon^j + B^2\gamma^k\delta^i p_1^j + AB\gamma^k p_2^i\varepsilon^j + B\gamma^k p_2^i p_1^j$$
$$+ ABp_3^k\delta^i\varepsilon^j + Bp_3^k\delta^i p_1^j + Ap_3^k p_2^i\varepsilon^j - 7p_3^k p_2^i p_1^j\big)\Big)\left(\frac{1}{\delta^i} - \frac{1}{\gamma^k}\right)$$

$$= \frac{1}{8}\sum_{1\le k\le r}\sum_{1\le i\le m_{23}}\sum_{1\le j\le m_{13}}\Bigg(\Bigg(AB^2\gamma^k\delta^i\varepsilon^j + B^2\gamma^k\delta^i\left(\frac{\varepsilon^j}{\alpha} - 1\right) + AB\gamma^k\left(\frac{\delta^i}{\alpha} - 1\right)\varepsilon^j$$
$$+ B\gamma^k\left(\frac{\delta^i}{\alpha} - 1\right)\left(\frac{\varepsilon^j}{\alpha} - 1\right) + AB\left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2}\right)\delta^i\varepsilon^j$$
$$+ B\left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2}\right)\delta^i\left(\frac{\varepsilon^j}{\alpha} - 1\right)$$
$$+ A\left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2}\right)\left(\frac{\delta^i}{\alpha} - 1\right)\varepsilon^j$$
$$- 7p_3^k\left(\frac{\delta^i}{\alpha} - 1\right)\left(\frac{\varepsilon^j}{\alpha} - 1\right)\Bigg)\Bigg)\left(\frac{1}{\delta^i} - \frac{1}{\gamma^k}\right)$$

$$= \frac{1}{8}\sum_{1\le k\le r}\sum_{1\le i\le m_{23}}\sum_{1\le j\le m_{13}}\Bigg(\Bigg(AB^2\gamma^k\delta^i\varepsilon^j + B^2\gamma^k\delta^i\left(\frac{\varepsilon^j}{\alpha} - 1\right) + AB\gamma^k\varepsilon^j\left(\frac{\delta^i}{\alpha} - 1\right)$$
$$+ B\gamma^k\left(\frac{\delta^i}{\alpha} - 1\right)\left(\frac{\varepsilon^j}{\alpha} - 1\right) + AB\delta^i\varepsilon^j\left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2}\right)$$
$$+ B\delta^i\left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2}\right)\left(\frac{\varepsilon^j}{\alpha} - 1\right)$$
$$+ A\varepsilon^j\left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2}\right)\left(\frac{\delta^i}{\alpha} - 1\right)$$
$$- 7\left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2}\right)\left(\frac{\delta^i}{\alpha} - 1\right)\left(\frac{\varepsilon^j}{\alpha} - 1\right)\Bigg)\Bigg)\left(\frac{1}{\delta^i} - \frac{1}{\gamma^k}\right)$$

$$\geq \frac{1}{8}\sum_{1\leq k\leq r}\sum_{1\leq i\leq m_{23}}\sum_{1\leq j\leq m_{13}}\Bigg(\Bigg(A^3\gamma^k\delta^i\varepsilon^j + A^2\gamma^k\delta^i\left(\frac{\varepsilon^j}{\alpha}-1\right) + A^2\gamma^k\varepsilon^j\left(\frac{\delta^i}{\alpha}-1\right)$$
$$+ A\gamma^k\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\varepsilon^j}{\alpha}-1\right) + A^2\delta^i\varepsilon^j\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)$$
$$+ A\delta^i\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\varepsilon^j}{\alpha}-1\right) + A\varepsilon^j\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\delta^i}{\alpha}-1\right)$$
$$- 7\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\varepsilon^j}{\alpha}-1\right)\Bigg)\Bigg)\left(\frac{1}{\delta^i}-\frac{1}{\gamma^k}\right)\ (A\varphi o\acute{\upsilon}\ B \geq A)$$

$$\geq \frac{1}{8}\sum_{1\leq k\leq r}\sum_{1\leq i\leq m_{23}}\sum_{1\leq j\leq m_{13}}\Bigg(\Bigg(A^3\gamma^k\delta^i\varepsilon^j + A^2\gamma^k\delta^i\left(\frac{\varepsilon^j}{\alpha}-1\right) + A^2\gamma^k\varepsilon^j\left(\frac{\delta^i}{\alpha}-1\right)$$
$$+ A\gamma^k\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\varepsilon^j}{\alpha}-1\right) + A^2\delta^i\varepsilon^j\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)$$
$$+ A\delta^i\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\varepsilon^j}{\alpha}-1\right) + A\varepsilon^j\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\delta^i}{\alpha}-1\right)$$
$$- 7\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\varepsilon^j}{\alpha}-1\right)\Bigg)\Bigg)\left(\frac{1}{\delta^i}-\frac{1}{\gamma^k}\right)$$

$$= \frac{1}{8}\sum_{1\leq k\leq r}\sum_{1\leq i\leq m_{23}}\sum_{1\leq j\leq m_{13}}\Bigg(A^3\gamma^k\delta^i\varepsilon^j\left(\frac{1}{\delta^i}-\frac{1}{\gamma^k}\right) + A^2\gamma^k\delta^i\left(\frac{\varepsilon^j}{\alpha}-1\right)\left(\frac{1}{\delta^i}-\frac{1}{\gamma^k}\right)$$
$$+ A^2\gamma^k\varepsilon^j\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{1}{\delta^i}-\frac{1}{\gamma^k}\right) + A\gamma^k\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\varepsilon^j}{\alpha}-1\right)\left(\frac{1}{\delta^i}-\frac{1}{\gamma^k}\right)$$
$$+ A^2\delta^i\varepsilon^j\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{1}{\delta^i}-\frac{1}{\gamma^k}\right)$$
$$+ A\delta^i\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\varepsilon^j}{\alpha}-1\right)\left(\frac{1}{\delta^i}-\frac{1}{\gamma^k}\right)$$
$$+ A\varepsilon^j\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{1}{\delta^i}-\frac{1}{\gamma^k}\right)$$
$$- 7\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\varepsilon^j}{\alpha}-1\right)\left(\frac{1}{\delta^i}-\frac{1}{\gamma^k}\right)\Bigg)$$

$$
\begin{aligned}
= \frac{1}{8} \sum_{1\le k\le r} \sum_{1\le i\le m_{23}} \sum_{1\le j\le m_{13}} \Bigg(& A^3\gamma^k\varepsilon^j\left(1-\frac{\delta^i}{\gamma^k}\right) + A^2\gamma^k\left(\frac{\varepsilon^j}{\alpha}-1\right)\left(1-\frac{\delta^i}{\gamma^k}\right) \\
& + A^2\varepsilon^j\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\gamma^k}{\delta^i}-1\right) + A\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\varepsilon^j}{\alpha}-1\right)\left(\frac{\gamma^k}{\delta^i}-1\right) \\
& + A^2\varepsilon^j\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(1-\frac{\delta^i}{\gamma^k}\right) \\
& + A\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\varepsilon^j}{\alpha}-1\right)\left(1-\frac{\delta^i}{\gamma^k}\right) \\
& + A\frac{\varepsilon^j}{\gamma^k}\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\gamma^k}{\delta^i}-1\right) \\
& - \frac{7}{\gamma^k}\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\varepsilon^j}{\alpha}-1\right)\left(\frac{\gamma^k}{\delta^i}-1\right)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
= \frac{1}{8} \sum_{1\le k\le r} \sum_{1\le i\le m_{23}} \sum_{1\le j\le m_{13}} \Bigg(& A^3\gamma^k\delta^i\varepsilon^j\left(1-\frac{\delta^i}{\gamma^k}\right) + A^2\gamma^k\delta^i\left(\frac{\varepsilon^j}{\alpha}-1\right)\left(1-\frac{\delta^i}{\gamma^k}\right) \\
& + A^2\varepsilon^j\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\gamma^k}{\delta^i}-1\right) + A\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\varepsilon^j}{\alpha}-1\right)\left(\frac{\gamma^k}{\delta^i}-1\right) \\
& + A^2\varepsilon^j\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(1-\frac{\delta^i}{\gamma^k}\right) \\
& + A\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\varepsilon^j}{\alpha}-1\right)\left(1-\frac{\delta^i}{\gamma^k}\right) \\
& + A\frac{\varepsilon^j}{\gamma^k}\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\gamma^k}{\delta^i}-1\right) \\
& - \frac{7}{\gamma^k}\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\varepsilon^j}{\alpha}-1\right)\left(\frac{\gamma^k}{\delta^i}-1\right)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
= \frac{1}{8} \sum_{1\le k\le r} \sum_{1\le i\le m_{23}} \sum_{1\le j\le m_{13}} \Bigg(& A^3\gamma^k\delta^i\varepsilon^j\left(1-\frac{\delta^i}{\gamma^k}\right) + A^2\gamma^k\delta^i\left(\frac{\varepsilon^j}{\alpha}-1\right)\left(1-\frac{\delta^i}{\gamma^k}\right) \\
& + A^2\varepsilon^j\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\gamma^k}{\delta^i}-1\right) + A\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\varepsilon^j}{\alpha}-1\right)\left(\frac{\gamma^k}{\delta^i}-1\right) \\
& + \left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\Bigg(A^2\varepsilon^j\left(1-\frac{\delta^i}{\gamma^k}\right) + A\left(\frac{\varepsilon^j}{\alpha}-1\right)\left(1-\frac{\delta^i}{\gamma^k}\right) \\
& + A\frac{\varepsilon^j}{\gamma^k}\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\gamma^k}{\delta^i}-1\right) - \frac{7}{\gamma^k}\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\varepsilon^j}{\alpha}-1\right)\left(\frac{\gamma^k}{\delta^i}-1\right)\Bigg)\Bigg)
\end{aligned}
$$

13. <u>Απλοποίηση της Δ_{13}</u>

$$\Delta 13 = \sum_{1\le k\le r} f^k \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le r} f^j\left(\frac{1}{\delta^i}-\frac{1}{\gamma^k}\right)$$

$$-\sum_{1\le k\le r} p_3^k \sum_{1\le i\le m_{23}} p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le r} p_1^j\left(\frac{1}{\delta^i}-\frac{1}{\gamma^k}\right)$$

$$= \sum_{1\le i\le m_{23}} \sum_{1\le k<j\le r} \left(f^k f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} f^j - p_3^k p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} p_1^j\right)\left(\frac{1}{\delta^i}-\frac{1}{\gamma^k}\right)$$

$$\ge \sum_{1\le i\le m_{23}} \sum_{1\le k<j\le r} \left(\frac{1}{8}\left(\gamma^k B + p_3^k\right)\left(B\delta^i + p_2^i\right)\left(\gamma^j B + p_1^j\right) - p_3^k p_2^i p_1^j\right)\left(\frac{1}{\delta^i}-\frac{1}{\gamma^k}\right)$$

$$= \sum_{1\le i\le m_{23}} \sum_{1\le k<j\le r} \left(\frac{1}{8}\left(\gamma^k B + p_3^k\right)\left(B\delta^i + p_2^i\right)\left(\gamma^j B + p_1^j\right) - \left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right) p_2^i \left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\right)\left(\frac{1}{\delta^i}-\frac{1}{\gamma^k}\right)$$

$$\begin{aligned} = \frac{1}{8} \sum_{1\le i\le m_{23}} \sum_{1\le k<j\le r} \Bigg(& B^3\gamma^k\gamma^j\delta^i\left(\frac{1}{\delta^i}-\frac{1}{\gamma^k}\right) + B^2\gamma^k\delta^i\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(\frac{1}{\delta^i}-\frac{1}{\gamma^k}\right) \\ & + B^2\gamma^k\gamma^j\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{1}{\delta^i}-\frac{1}{\gamma^k}\right) \\ & + B\gamma^k\left(\frac{\delta^j}{\alpha}-1\right)\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(\frac{1}{\delta^i}-\frac{1}{\gamma^k}\right) \\ & + B^2\delta^i\gamma^j\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{1}{\delta^i}-\frac{1}{\gamma^k}\right) \\ & + B\delta^i\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(\frac{1}{\delta^i}-\frac{1}{\gamma^k}\right) \\ & + B\gamma^j\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{1}{\delta^i}-\frac{1}{\gamma^k}\right) \\ & - 7\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(\frac{1}{\delta^i}-\frac{1}{\gamma^k}\right)\Bigg) \end{aligned}$$

$$
\begin{aligned}
= \frac{1}{8} \sum_{1\le i\le m_{23}} \sum_{1\le k<j\le r} \Bigg(& B^3\gamma^k\gamma^j\left(1-\frac{\delta^i}{\gamma^k}\right) + B^2\gamma^k\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(1-\frac{\delta^i}{\gamma^k}\right) \\
& + B^2\gamma^j\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\gamma^k}{\delta^i}-1\right) \\
& + B\left(\frac{\delta^j}{\alpha}-1\right)\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(\frac{\gamma^k}{\delta^i}-1\right) \\
& + B^2\gamma^j\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(1-\frac{\delta^i}{\gamma^k}\right) \\
& + B\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(1-\frac{\delta^i}{\gamma^k}\right) \\
& + \frac{B\gamma^j}{\gamma^k}\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\gamma^k}{\delta^i}-1\right) \\
& - \frac{7}{\gamma^k}\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(\frac{\gamma^k}{\delta^i}-1\right)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
= \frac{1}{8} \sum_{1\le i\le m_{23}} \sum_{1\le j\le r} \Bigg(\Bigg(& B^3\gamma^j\gamma^j + B^2\gamma^j\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right) + B^2\gamma^j\left(\frac{\delta^i}{\alpha}-1\right) \\
& + B\left(\frac{\delta^j}{\alpha}-1\right)\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right) + B^2\gamma^j\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right) \\
& + B\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right) \\
& + \frac{B\gamma^j}{\gamma^j}\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(\frac{\delta^i}{\alpha}-1\right) \\
& - \frac{7}{\gamma^j}\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(\frac{\delta^i}{\alpha}-1\right)\Bigg(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma} \\
& -\frac{1}{2}\Bigg)\Bigg)\frac{1}{\delta^i}\sum_{1\le k\le r}\left|\gamma^k-\delta^i\right|\Bigg)
\end{aligned}
$$

$$\geq \frac{1}{8}\sum_{1\leq i\leq m_{23}}\sum_{1\leq j\leq r}\left(\left(B^2\gamma^j\left(B\gamma^j+\left(\frac{\delta^i}{\alpha}-1\right)\right)\right.\right.$$
$$+\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(2B^2\gamma^j+B\left(\frac{\delta^j}{\alpha}-1\right)\right.$$
$$+B\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)+B\left(\frac{\delta^i}{\alpha}-1\right)$$
$$\left.\left.\left.-\frac{7}{\gamma^j}\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\right)\right)\frac{1}{\delta^i}S(i)\right)\left(A\varphi o\acute{\upsilon}\ \left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}\right.\right.$$
$$\left.\left.+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\leq\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\right)$$

14. Απλοποίηση της Δ_{14}

$$\Delta 14=\sum_{1\leq k\leq r}f^k\sum_{1\leq i\leq r}f^i\sum_{1\leq j\leq r}f^j\left(\frac{1}{\gamma^i}-\frac{1}{\gamma^k}\right)-\sum_{1\leq k\leq r}p_3^k\sum_{1\leq i\leq r}p_2^i\sum_{1\leq j\leq r}p_1^j\left(\frac{1}{\gamma^i}-\frac{1}{\gamma^k}\right)$$

$$=\sum_{1\leq k<j<i\leq r}\left(f^kf^if^j-p_3^kp_2^ip_1^j\right)\left(\frac{1}{\gamma^i}-\frac{1}{\gamma^k}\right)$$

$$=\frac{1}{8}\sum_{1\leq k<j<i\leq r}\left(\left(\gamma^kB+\frac{\gamma^k}{\beta}-1\right)\left(\gamma^iA+\frac{\gamma^i}{a}-1\right)\left(\gamma^jB+\frac{\gamma^j}{\beta}-1\right)\right.$$
$$\left.-\left(\frac{\gamma^k}{\beta}-\frac{\gamma^k}{\alpha}+\frac{\gamma^k}{\gamma}-1\right)\left(\frac{\gamma^i}{a}-\frac{\gamma^i}{\beta}+\frac{\gamma^i}{\gamma}-1\right)\left(\frac{\gamma^j}{\beta}+\frac{\gamma^j}{\alpha}-\frac{\gamma^j}{\gamma}-1\right)\right)\left(\frac{1}{\gamma^i}-\frac{1}{\gamma^k}\right)$$

$$=\frac{1}{8}\sum_{1\leq k<j<i\leq r}\left(\left(\gamma^kB+\frac{\gamma^k}{\beta}-1\right)\left(\gamma^iA+\frac{\gamma^i}{a}-1\right)\left(\gamma^jB+\frac{\gamma^j}{\beta}-1\right)\right.$$
$$\left.-\left(\frac{\gamma^k}{\beta}-\frac{\gamma^k}{\alpha}+\frac{\gamma^k}{\gamma}-1\right)\left(\frac{\gamma^i}{a}-\frac{\gamma^i}{\beta}+\frac{\gamma^i}{\gamma}-1\right)\left(\frac{\gamma^j}{\beta}+\frac{\gamma^j}{\alpha}-\frac{\gamma^j}{\gamma}-1\right)\right)\left(\frac{1}{\gamma^i}-\frac{1}{\gamma^k}\right)$$

$$\geq \frac{1}{8}\sum_{1\leq k<j<i\leq r}\Bigg(\left(\left(\gamma^k B+\frac{\gamma^k}{\beta}-1\right)\left(\gamma^i A+\frac{\gamma^i}{a}-1\right)\left(\gamma^j B+\frac{\gamma^j}{\beta}-1\right)\right.$$

$$\left.-\left(\frac{\gamma^k}{\beta}-1\right)\left(\frac{\gamma^i}{a}-1\right)\left(\frac{\gamma^j}{\beta}-1\right)\right)\left(\frac{1}{\gamma^i}-\frac{1}{\gamma^k}\right)(A\varphi o\acute{\upsilon}\ \gamma\geq\beta\geq\alpha)$$

15. <u>Απλοποίηση της Δ_{15}</u>

$$\Delta 15=\sum_{1\leq k\leq m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k}\sum_{1\leq i\leq m_2} f^{r+m_1+i}\sum_{1\leq j\leq m_{12}} f^{r+m_1+m_2+m_3+j}\left(\frac{1}{\beta}-\frac{1}{\varepsilon^k}\right)$$

$$-\sum_{1\leq k\leq m_{13}} p_3^{r+m_1+m_2+m_3+m_{12}+k}\sum_{1\leq i\leq m_2} p_2^{r+m_1+i}\sum_{1\leq j\leq m_{12}} p_1^{r+m_1+m_2+m_3+j}\left(\frac{1}{\beta}-\frac{1}{\varepsilon^k}\right)$$

$$=\sum_{1\leq k\leq m_{13}}\sum_{1\leq i\leq m_2}\sum_{1\leq j\leq m_{12}} f^{r+m_1+m_2+m_3+m_{12}+k} f^{r+m_1+i} f^{r+m_1+m_2+m_3+j}\left(\frac{1}{\beta}-\frac{1}{\varepsilon^k}\right)$$

$$-\sum_{1\leq k\leq m_{13}}\sum_{1\leq i\leq m_2}\sum_{1\leq j\leq m_{12}} p_3^{r+m_1+m_2+m_3+m_{12}+k} p_2^{r+m_1+i} p_1^{r+m_1+m_2+m_3+j}\left(\frac{1}{\beta}-\frac{1}{\varepsilon^k}\right)$$

$$=\sum_{1\leq k\leq m_{13}}\sum_{1\leq j\leq m_{12}}\Bigg(f^{r+m_1+m_2+m_3+m_{12}+k} m_2 f^{r+m_1+1} f^{r+m_1+m_2+m_3+j}$$

$$-p_3^{r+m_1+m_2+m_3+m_{12}+k}\left(\frac{(r+2)\alpha-\Gamma}{2\alpha}-\frac{Z-m_{12}\gamma}{\gamma}-\frac{\Delta-m_{23}\alpha}{\alpha}\right.$$

$$\left.+\frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)p_1^{r+m_1+m_2+m_3+j}\Bigg)\left(\frac{1}{\beta}-\frac{1}{\varepsilon^k}\right)$$

$=\sum_{1\leq k\leq m_{13}}\sum_{1\leq j\leq m_{12}}\Bigg(\frac{1}{2}\left(A\varepsilon^k+\frac{\varepsilon^k}{\gamma}-1\right)m_2\beta B\,\frac{1}{2}\left(B\zeta^j+\frac{\zeta^j}{\beta}-1\right)-\left(\frac{\varepsilon^k}{\gamma}-1\right)\left(\frac{(r+2)\alpha-\Gamma}{2\alpha}-\frac{Z-m_{12}\gamma}{\gamma}-\right.$
$\left.\frac{\Delta-m_{23}\alpha}{\alpha}+\frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\left(\frac{\zeta^j}{\beta}-1\right)\Bigg)\left(\frac{1}{\beta}-\frac{1}{\varepsilon^k}\right)$

$=\left(\frac{m_2\beta B}{4}-\left(\frac{(r+2)\alpha-\Gamma}{2\alpha}-\frac{Z-m_{12}\gamma}{\gamma}-\frac{\Delta-m_{23}\alpha}{\alpha}+\frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\right)\frac{1}{\beta^2\gamma}\sum_{1\leq k\leq m_{13}}\sum_{1\leq j\leq m_{12}}\left((\varepsilon^k-\gamma)(\zeta^j-\beta)\ \frac{(\varepsilon^k-\beta)}{\varepsilon^k}\right)$

16. Απλοποίηση της $\varDelta_{16}$

$$\varDelta 16=\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k}\sum_{1\le i\le m_2} f^{r+m_1+i}\sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\beta}-\frac{1}{\varepsilon^k}\right)$$

$$-\sum_{1\le k\le m_{13}} p_3^{r+m_1+m_2+m_3+m_{12}+k}\sum_{1\le i\le m_2} p_2^{r+m_1+i}\sum_{1\le j\le m_{13}} p_1^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\beta}-\frac{1}{\varepsilon^k}\right)$$

$$=\sum_{1\le i\le m_2}\sum_{1\le j<k\le m_{13}}\left(f^{r+m_1+m_2+m_3+m_{12}+k}f^{r+m_1+i}f^{r+m_1+m_2+m_3+m_{12}+j}\right.$$
$$\left.-p_3^{r+m_1+m_2+m_3+m_{12}+k}p_2^{r+m_1+i}p_1^{r+m_1+m_2+m_3+m_{12}+j}\right)\left(\frac{1}{\beta}-\frac{1}{\varepsilon^k}\right)$$

$$=\sum_{1\le j<k\le m_{13}}\left(m_2\beta B\frac{1}{2}\left(A\varepsilon^k+\frac{\varepsilon^k}{\gamma}-1\right)\frac{1}{2}\left(A\varepsilon^j+\frac{\varepsilon^j}{a}-1\right)\right.$$
$$-\left(\frac{(r+2)\alpha-\varGamma}{2\alpha}-\frac{Z-m_{12}\gamma}{\gamma}-\frac{\varDelta-m_{23}\alpha}{\alpha}+\frac{\varGamma(\gamma-\beta)}{2\beta\gamma}\right)\left(\frac{\varepsilon^k}{\gamma}-1\right)\left(\frac{\varepsilon^j}{\alpha}\right.$$
$$\left.\left.-1\right)\right)\left(\frac{1}{\beta}-\frac{1}{\varepsilon^k}\right)$$

$$=\sum_{1\le j<k\le m_{13}}\left(m_2\beta B\frac{1}{4}\left(A^2\varepsilon^j\varepsilon^k+\frac{A\varepsilon^j\varepsilon^k}{a}+\frac{A\varepsilon^j\varepsilon^k}{\gamma}-A\varepsilon^k-A\varepsilon^j+\frac{\varepsilon^j\varepsilon^k}{a\gamma}-\frac{\varepsilon^k}{\gamma}-\frac{\varepsilon^j}{a}+1\right)\right.$$
$$-\left(\frac{(r+2)\alpha-\varGamma}{2\alpha}-\frac{Z-m_{12}\gamma}{\gamma}-\frac{\varDelta-m_{23}\alpha}{\alpha}+\frac{\varGamma(\gamma-\beta)}{2\beta\gamma}\right)\left(\frac{\varepsilon^j\varepsilon^k}{a\gamma}-\frac{\varepsilon^k}{\gamma}-\frac{\varepsilon^j}{\alpha}\right.$$
$$\left.\left.+1\right)\right)\left(\frac{1}{\beta}-\frac{1}{\varepsilon^k}\right)$$

$=$
$m_2\beta B\frac{1}{4}\sum_{1\le j<k\le m_{13}}\left(A^2\varepsilon^j\varepsilon^k+\frac{A\varepsilon^j\varepsilon^k}{a}+\frac{A\varepsilon^j\varepsilon^k}{\gamma}-A\varepsilon^k-A\varepsilon^j\right)\left(\frac{1}{\beta}-\frac{1}{\varepsilon^k}\right)+$
$m_2\beta B\frac{1}{4}\sum_{1\le j<k\le m_{13}}\left(\frac{\varepsilon^j\varepsilon^k}{a\gamma}-\frac{\varepsilon^k}{\gamma}-\frac{\varepsilon^j}{a}+1\right)\left(\frac{1}{\beta}-\frac{1}{\varepsilon^k}\right)-\left(m_{12}+m_{13}+\frac{\varGamma+\beta(r+2)}{2\beta}-\frac{\varGamma+2Z}{2\gamma}-\right.$
$\left.\frac{\varGamma+2\varDelta}{2a}\right)\sum_{1\le j<k\le m_{13}}\left(\frac{\varepsilon^j\varepsilon^k}{a\gamma}-\frac{\varepsilon^k}{\gamma}-\frac{\varepsilon^j}{\alpha}+1\right)\left(\frac{1}{\beta}-\frac{1}{\varepsilon^k}\right)$

$=m_2\beta AB\frac{1}{4}\sum_{1\le j<k\le m_{13}}\left(A\varepsilon^j\varepsilon^k+\frac{\varepsilon^j\varepsilon^k}{a}+\frac{\varepsilon^j\varepsilon^k}{\gamma}-\varepsilon^k-\varepsilon^j\right)\left(\frac{1}{\beta}-\frac{1}{\varepsilon^k}\right)+\left(m_2\beta B\frac{1}{4}-\left(\frac{(r+2)\alpha-\varGamma}{2\alpha}-\right.\right.$
$\left.\left.\frac{Z-m_{12}\gamma}{\gamma}-\frac{\varDelta-m_{23}\alpha}{\alpha}+\frac{\varGamma(\gamma-\beta)}{2\beta\gamma}\right)\right)\sum_{1\le j<k\le m_{13}}\left(\frac{\varepsilon^j\varepsilon^k}{a\gamma}-\frac{\varepsilon^k}{\gamma}-\frac{\varepsilon^j}{a}+1\right)\left(\frac{1}{\beta}-\frac{1}{\varepsilon^k}\right)$

$$\geq \left(\frac{m_2\beta B}{4} - \left(\frac{(r+2)\alpha-\Gamma}{2\alpha} - \frac{Z-m_{12}\gamma}{\gamma} - \frac{\Delta-m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\right)\sum_{1\leq j<k\leq m_{13}}\left(\left(\frac{\varepsilon^k}{\gamma}-1\right)\left(\frac{\varepsilon^j}{a}-1\right)\left(\frac{1}{\beta}-\frac{1}{\varepsilon^k}\right)\right)$$

$$= \left(\frac{m_2\beta B}{4} - \left(\frac{(r+2)\alpha-\Gamma}{2\alpha} - \frac{Z-m_{12}\gamma}{\gamma} - \frac{\Delta-m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\right)\frac{1}{\alpha\beta\gamma}\sum_{1\leq j<k\leq m_{13}}\left((\varepsilon^k-\gamma)(\varepsilon^j - \alpha)\left(\frac{\varepsilon^k-\beta}{\varepsilon^k}\right)\right)$$

$$\geq \left(\frac{m_2\beta B}{4} - \left(\frac{(r+2)\alpha-\Gamma}{2\alpha} - \frac{Z-m_{12}\gamma}{\gamma} - \frac{\Delta-m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\right)\frac{1}{\alpha\beta\gamma}\sum_{1\leq j<k\leq m_{13}}\left((\varepsilon^j - \alpha)\left(\frac{(\varepsilon^k-\gamma)^2}{\varepsilon^k}\right)\right)$$

(Αφού $\gamma \geq \beta$)

17. <u>Απλοποίηση της Δ_{17}</u>

$$\Delta 17 = \sum_{1\leq k\leq m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\leq i\leq m_2} f^{r+m_1+i} \sum_{1\leq j\leq r} f^j\left(\frac{1}{\beta}-\frac{1}{\varepsilon^k}\right)$$
$$- \sum_{1\leq k\leq m_{13}} p_3^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\leq i\leq m_2} p_2^{r+m_1+i} \sum_{1\leq j\leq r} p_1^j\left(\frac{1}{\beta}-\frac{1}{\varepsilon^k}\right)$$
$$= \sum_{1\leq k\leq m_{13}}\sum_{1\leq i\leq m_2}\sum_{1\leq j\leq r}\left(f^{r+m_1+m_2+m_3+m_{12}+k} f^{r+m_1+i} f^j - p_3^{r+m_1+m_2+m_3+m_{12}+k} p_2^{r+m_1+i} p_1^j\right)\left(\frac{1}{\beta}-\frac{1}{\varepsilon^k}\right)$$

$$= \sum_{1\leq k\leq m_{13}}\sum_{1\leq j\leq r}\left(\frac{1}{2}\left(A\varepsilon^k + \frac{\varepsilon^k}{\gamma} - 1\right) m_2\beta B \frac{1}{2}\left(\gamma^j B + \frac{\gamma^j}{\beta} - 1\right) - \left(\frac{\varepsilon^k}{\gamma}-1\right)\left(\frac{(r+2)\alpha-\Gamma}{2\alpha} - \frac{Z-m_{12}\gamma}{\gamma} - \frac{\Delta-m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\right)\left(\frac{1}{\beta}-\frac{1}{\varepsilon^k}\right)$$

Έχουμε ότι $p_1^j + p_3^j = \left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right) + \left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right) = \frac{\gamma^j}{\beta} - 1$, άρα $\frac{\gamma^j}{\beta} - 1 \geq \frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}$, για όλες τις συνδέσεις j ∈ [r], επομένως,

$$\geq \left(\frac{m_2\beta B}{4} - \left(\frac{(r+2)\alpha-\Gamma}{2\alpha} - \frac{Z-m_{12}\gamma}{\gamma} - \frac{\Delta-m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\right)\sum_{1\leq k\leq m_{13}}\sum_{1\leq j\leq r}\left(\left(\frac{\varepsilon^k}{\gamma}-1\right)\left(\frac{\gamma^j}{\beta}-1\right)\left(\frac{1}{\beta}-\frac{1}{\varepsilon^k}\right)\right)$$

$$= \left(\frac{m_2\beta B}{4} - \left(\frac{(r+2)\alpha - \Gamma}{2\alpha} - \frac{Z - m_{12}\gamma}{\gamma} - \frac{\Delta - m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\right)\frac{1}{\beta^2\gamma}\sum_{1\le k\le m_{13}}\sum_{1\le j\le r}\left(\left(\varepsilon^k - \gamma\right)\left(\gamma^j - \beta\right)\frac{\left(\varepsilon^k - \beta\right)}{\varepsilon^k}\right)$$

$$\ge \left(\frac{m_2\beta B}{4} - \left(\frac{(r+2)\alpha - \Gamma}{2\alpha} - \frac{Z - m_{12}\gamma}{\gamma} - \frac{\Delta - m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\right)\frac{1}{\beta^2\gamma}\sum_{1\le k\le m_{13}}\sum_{1\le j\le r}\left(\left(\gamma^j - \beta\right)\frac{\left(\varepsilon^k - \gamma\right)^2}{\varepsilon^k}\right)$$

18. Απλοποίηση της Δ_{18}

$$\Delta 18 = \sum_{1\le k\le m_{23}} f^{+m_1+m_2+m_3+m_{12}+m_{13}+k}\sum_{1\le i\le m_2} f^{r+m_1+i}\sum_{1\le j\le r} f^j\left(\frac{1}{\beta} - \frac{1}{\delta^k}\right) - \sum_{1\le k\le m_{23}} p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k}\sum_{1\le i\le m_2} p_2^{r+m_1+i}\sum_{1\le j\le r} p_1^j\left(\frac{1}{\beta} - \frac{1}{\delta^k}\right)$$

$$= \sum_{1\le k\le m_{23}}\sum_{1\le i\le m_2}\sum_{1\le j\le r} f^{+m_1+m_2+m_3+m_{12}+m_{13}+k}f^{r+m_1+i}f^j\left(\frac{1}{\beta} - \frac{1}{\delta^k}\right) - \sum_{1\le k\le m_{23}}\sum_{1\le i\le m_2}\sum_{1\le j\le r} p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k}p_2^{r+m_1+i}p_1^j\left(\frac{1}{\beta} - \frac{1}{\delta^k}\right)$$

$$= \sum_{1\le k\le m_{23}}\sum_{1\le i\le m_2}\sum_{1\le j\le r}\left(f^{+m_1+m_2+m_3+m_{12}+m_{13}+k}f^{r+m_1+i}f^j - p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k}p_2^{r+m_1+i}p_1^j\right)\left(\frac{1}{\beta} - \frac{1}{\delta^k}\right)$$

$$= \sum_{1\le k\le m_{23}}\sum_{1\le j\le r}\left(\frac{1}{2}\left(\delta^k B + \frac{\delta^k}{\beta} - 1\right)m_2\beta B\frac{1}{2}\left(\gamma^j B + \frac{\gamma^j}{\beta} - 1\right) - \left(\frac{\delta^k}{\beta} - 1\right)\left(\frac{(r+2)\alpha - \Gamma}{2\alpha} - \frac{Z - m_{12}\gamma}{\gamma} - \frac{\Delta - m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\left(\frac{\gamma^j}{2\beta} + \frac{\gamma^j}{2\alpha} - \frac{\gamma^j}{2\gamma} - \frac{1}{2}\right)\right)\left(\frac{1}{\beta} - \frac{1}{\varepsilon^k}\right)$$

$$\ge \left(\frac{m_2\beta B}{4} - \left(\frac{(r+2)\alpha - \Gamma}{2\alpha} - \frac{Z - m_{12}\gamma}{\gamma} - \frac{\Delta - m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\right)\frac{1}{\beta^3}\sum_{1\le k\le m_{23}}\sum_{1\le j\le r}\left(\left(\delta^k - \beta\right)\left(\gamma^j - \beta\right)\frac{\left(\varepsilon^k - \beta\right)}{\varepsilon^k}\right)$$

$$\ge \left(\frac{m_2\beta B}{4} - \left(\frac{(r+2)\alpha - \Gamma}{2\alpha} - \frac{Z - m_{12}\gamma}{\gamma} - \frac{\Delta - m_{23}\alpha}{\alpha} + \frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\right)\frac{1}{\beta^3}\sum_{1\le k\le m_{23}}\sum_{1\le j\le r}\left(\left(\delta^k - \beta\right)\left(\gamma^j - \beta\right)\frac{\left(\varepsilon^k - \gamma\right)}{\varepsilon^k}\right)$$

19. <u>Απλοποίηση της $Δ_{19}$</u>

$$Δ19=\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k}\sum_{1\le i\le m_2} f^{r+m_1+i}\sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j}\left(\frac{1}{\beta}-\frac{1}{\delta^k}\right)$$

$$-\sum_{1\le k\le m_{23}} p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k}\sum_{1\le i\le m_2} p_2^{r+m_1+i}\sum_{1\le j\le m_{12}} p_1^{r+m_1+m_2+m_3+j}\left(\frac{1}{\beta}-\frac{1}{\delta^k}\right)$$

$$=\sum_{1\le k\le m_{23}}\sum_{1\le i\le m_2}\sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} f^{r+m_1+i} f^{r+m_1+m_2+m_3+j}\left(\frac{1}{\beta}-\frac{1}{\delta^k}\right)$$

$$-\sum_{1\le k\le m_{23}}\sum_{1\le i\le m_2}\sum_{1\le j\le m_{12}} p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} p_2^{r+m_1+i} p_1^{r+m_1+m_2+m_3+j}\left(\frac{1}{\beta}-\frac{1}{\delta^k}\right)$$

$$=\sum_{1\le k\le m_{23}}\sum_{1\le i\le m_2}\sum_{1\le j\le m_{12}}\Big(f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} f^{r+m_1+i} f^{r+m_1+m_2+m_3+j}$$
$$-p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} p_2^{r+m_1+i} p_1^{r+m_1+m_2+m_3+j}\Big)\left(\frac{1}{\beta}-\frac{1}{\delta^k}\right)$$

$=\sum_{1\le k\le m_{23}}\sum_{1\le j\le m_{12}}\left(\frac{1}{2}\left(\delta^k B+\frac{\delta^k}{\beta}-1\right)m_2\beta B\frac{1}{2}\left(B\zeta^j+\frac{\zeta^j}{\beta}-1\right)-\left(\frac{\delta^k}{\beta}-1\right)\left(m_{12}+m_{13}+\right.\right.$
$\left.\left.\frac{\Gamma+\beta(r+2)}{2\beta}-\frac{\Gamma+2Z}{2\gamma}-\frac{\Gamma+2\Delta}{2\alpha}\right)\left(\frac{\zeta^j}{\beta}-1\right)\right)\left(\frac{1}{\beta}-\frac{1}{\delta^k}\right)$

$\ge\left(\frac{m_2\beta B}{4}-\left(\frac{(r+2)\alpha-\Gamma}{2\alpha}-\frac{Z-m_{12}\gamma}{\gamma}-\frac{\Delta-m_{23}\alpha}{\alpha}+\frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\right)\frac{1}{\beta^3}\sum_{1\le k\le m_{23}}\sum_{1\le j\le m_{12}}\left(\left(\delta^k-\right.\right.$
$\left.\left.\beta\right)\left(\zeta^j-\beta\right)\frac{\left(\delta^k-\beta\right)}{\delta^k}\right)$

$\ge\left(\frac{m_2\beta B}{4}-\left(\frac{(r+2)\alpha-\Gamma}{2\alpha}-\frac{Z-m_{12}\gamma}{\gamma}-\frac{\Delta-m_{23}\alpha}{\alpha}+\frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\right)\frac{1}{\beta^3}\sum_{1\le k\le m_{23}}\sum_{1\le j\le m_{12}}\left(\left(\delta^k-\right.\right.$
$\left.\left.\beta\right)\left(\zeta^j-\beta\right)\frac{\left(\delta^k-\gamma\right)}{\delta^k}\right)$

20. <u>Απλοποίηση της $\varDelta_{20}$</u>

$$\varDelta 20 = \sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\beta}-\frac{1}{\delta^k}\right)$$
$$-\sum_{1\le k\le m_{23}} p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_2} p_2^{r+m_1+i} \sum_{1\le j\le m_{13}} p_1^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\beta}-\frac{1}{\delta^k}\right)$$
$$=\sum_{1\le k\le m_{23}}\sum_{1\le i\le m_2}\sum_{1\le j\le m_{13}}\left(f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} f^{r+m_1+i} f^{r+m_1+m_2+m_3+m_{12}+j}\right.$$
$$\left.-p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} p_2^{r+m_1+i} p_1^{r+m_1+m_2+m_3+m_{12}+j}\right)\left(\frac{1}{\beta}-\frac{1}{\delta^k}\right)$$

$$=\sum_{1\le k\le m_{23}}\sum_{1\le j\le m_{13}}\left(\frac{1}{2}\left(\delta^k B+\frac{\delta^k}{\beta}-1\right)m_2\beta B\,\frac{1}{2}\left(A\varepsilon^j+\frac{\varepsilon^j}{\alpha}-1\right)-\left(\frac{\delta^k}{\beta}-1\right)\left(\frac{(r+2)\alpha-\Gamma}{2\alpha}-\frac{Z-m_{12}\gamma}{\gamma}-\right.\right.$$
$$\left.\left.\frac{\varDelta-m_{23}\alpha}{\alpha}+\frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\left(\frac{\varepsilon^j}{\alpha}-1\right)\right)\left(\frac{1}{\beta}-\frac{1}{\delta^k}\right)$$
$$\ge\left(\frac{m_2\beta B}{4}-\left(\frac{(r+2)\alpha-\Gamma}{2\alpha}-\frac{Z-m_{12}\gamma}{\gamma}-\frac{\varDelta-m_{23}\alpha}{\alpha}+\frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\right)\frac{1}{\alpha\beta^2}\sum_{1\le k\le m_{23}}\sum_{1\le j\le m_{13}}\left((\delta^k-\beta)(\varepsilon^j-\alpha)\frac{(\delta^k-\beta)}{\delta^k}\right)$$
$$\ge\left(\frac{m_2\beta B}{4}-\left(\frac{(r+2)\alpha-\Gamma}{2\alpha}-\frac{Z-m_{12}\gamma}{\gamma}-\frac{\varDelta-m_{23}\alpha}{\alpha}+\frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\right)\frac{1}{\alpha\beta^2}\sum_{1\le k\le m_{23}}\sum_{1\le j\le m_{13}}\left((\delta^k-\beta)(\varepsilon^j-\alpha)\frac{(\delta^k-\gamma)}{\delta^k}\right)$$

21. <u>Απλοποίηση της $\varDelta_{21}$</u>

$$\varDelta 21 = \sum_{1\le k\le r} f^k \sum_{1\le i\le m_2} f^{r+m_1+i} \sum_{1\le j\le r} f^j\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)$$
$$-\sum_{1\le k\le r} p_3^k \sum_{1\le i\le m_2} p_2^{r+m_1+i} \sum_{1\le j\le r} p_1^j\left(\frac{1}{\beta}-\frac{1}{\delta^k}\right)$$

$$=\sum_{1\le i\le m_2}\sum_{1\le k<j\le r}\left(f^k f^{r+m_1+i} f^j-p_3^k p_2^{r+m_1+i} p_1^j\right)\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)$$

$$=\sum_{1\le k<j\le r}\left(\frac{1}{2}\left(\gamma^k B+\frac{\gamma^k}{\beta}-1\right)m_2\beta B\,\frac{1}{2}\left(\gamma^j B+\frac{\gamma^j}{\beta}-1\right)-\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{(r+2)\alpha-\Gamma}{2\alpha}-\frac{Z-m_{12}\gamma}{\gamma}-\right.\right.$$
$$\left.\left.\frac{\varDelta-m_{23}\alpha}{\alpha}+\frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\right)\left(\frac{1}{\beta}-\frac{1}{\gamma^k}\right)$$

$$\geq \left(\frac{m_2\beta B}{4}-\left(\frac{(r+2)\alpha-\Gamma}{2\alpha}-\frac{Z-m_{12}\gamma}{\gamma}-\frac{\Delta-m_{23}\alpha}{\alpha}+\frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\right)\frac{1}{\beta^3}\sum_{1\leq k<j\leq r}\left(\left(\gamma^k-\beta\right)\left(\gamma^j-\beta\right)\frac{(\gamma^k-\beta)}{\gamma^k}\right)$$

$$\geq \left(\frac{m_2\beta B}{4}-\left(\frac{(r+2)\alpha-\Gamma}{2\alpha}-\frac{Z-m_{12}\gamma}{\gamma}-\frac{\Delta-m_{23}\alpha}{\alpha}+\frac{\Gamma(\gamma-\beta)}{2\beta\gamma}\right)\right)\frac{1}{\beta^3}\sum_{1\leq k<j\leq r}\left(\left(\gamma^k-\beta\right)\left(\gamma^j-\beta\right)\frac{(\gamma^k-\gamma)}{\gamma^k}\right)$$

22. <u>Απλοποίηση της Δ_{22}</u>

$$\Delta 22 = \sum_{1\leq k\leq r} f^k \sum_{1\leq i\leq m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\leq j\leq r} f^j\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^k}\right) - \sum_{1\leq k\leq r} p_3^k \sum_{1\leq i\leq m_{12}} p_2^{r+m_1+m_2+m_3+i} \sum_{1\leq j\leq r} p_1^j\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^k}\right)$$

$$= \sum_{1\leq k\leq r}\sum_{1\leq i\leq m_{12}}\sum_{1\leq j\leq r}\left(f^k f^{r+m_1+m_2+m_3+i} f^j - p_3^k p_2^{r+m_1+m_2+m_3+i} p_1^j\right)\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^k}\right)$$

$$= \sum_{1\leq i\leq m_{12}}\sum_{1\leq k<j\leq r}\left(\frac{1}{2}\left(\gamma^k B+\frac{\gamma^k}{\beta}-1\right)\frac{1}{2}\left(G\zeta^i+\frac{\zeta^i}{\gamma}-1\right)\frac{1}{2}\left(\gamma^j B+\frac{\gamma^j}{\beta}-1\right)-\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\right)\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^k}\right)$$

$$\geq \sum_{1\leq i\leq m_{12}}\sum_{1\leq k<j\leq r}\left(\frac{1}{2}\left(B\gamma^k+p_3^k\right)\frac{1}{2}\left(G\zeta^i+p_2^i\right)\frac{1}{2}\left(B\gamma^j+p_1^j\right)-p_3^k p_2^i p_1^j\right)\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^k}\right)$$

$$= \sum_{1\leq i\leq m_{12}}\sum_{1\leq k<j\leq r}\left(\frac{1}{8}\left(B^2G{\gamma^k}\zeta^i\gamma^j+BG\gamma^k\zeta^i p_1^j+B^2{\gamma^k}\gamma^j p_2^i+B\gamma^k p_2^i p_1^j+BG\gamma^j\zeta^i p_3^k + Gp_3^k p_1^j\zeta^i+B\gamma^j p_3^k p_2^i-7p_3^k p_2^i p_1^j\right)\right)\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^k}\right)$$

$$\geq \sum_{1\leq i\leq m_{12}}\sum_{1\leq k<j\leq r}\left(\frac{1}{8}\left(B^3\gamma^k\zeta^i\gamma^j+B^2\gamma^k\zeta^i p_1^j+B^2\gamma^k\gamma^j p_2^i+B\gamma^k p_2^i p_1^j+B^2{\gamma^j}\zeta^i p_3^k + Bp_3^k p_1^j\zeta^i+B\gamma^j p_3^k p_2^i-7p_3^k p_2^i p_1^j\right)\right)\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^k}\right)(\text{Αφού } G\geq B)$$

$$
\begin{aligned}
&= \sum_{1\le i\le m_{12}} \sum_{1\le k<j\le r} \Bigg(\frac{1}{8}\Bigg(B^3\gamma^k\gamma^j\left(1-\frac{\zeta^i}{\gamma^k}\right)+B^2\gamma^k\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(1-\frac{\zeta^i}{\gamma^k}\right)\\
&\qquad +B^2\gamma^j\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{\gamma^k}{\zeta^i}-1\right)+B\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(\frac{\gamma^k}{\zeta^i}-1\right)\\
&\qquad +B^2\gamma^j\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(1-\frac{\zeta^i}{\gamma^k}\right)\\
&\qquad +B\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(1-\frac{\zeta^i}{\gamma^k}\right)\\
&\qquad +\frac{B\gamma^j}{\gamma^k}\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{\gamma^k}{\zeta^i}-1\right)\\
&\qquad -\frac{7}{\gamma^k}\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(\frac{\gamma^k}{\zeta^i}-1\right)\Bigg)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&=\frac{1}{8}\sum_{1\le i\le m_{12}} \sum_{1\le j\le r} \Bigg(\Bigg(B^3\gamma^j\gamma^j+B^2\gamma^j\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)+B^2\gamma^j\left(\frac{\zeta^i}{\gamma}-1\right)\\
&\qquad +B\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)+B^2\gamma^j\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\\
&\qquad +B\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)+\frac{B\gamma^j}{\gamma^j}\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(\frac{\zeta^i}{\gamma}-1\right)\\
&\qquad -\frac{7}{\gamma^j}\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\Bigg)\sum_{1\le k\le r}\left|1-\frac{\zeta^i}{\gamma^k}\right|\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&\ge\frac{1}{8}\sum_{1\le i\le m_{12}} \sum_{1\le j\le r} \Bigg(\Bigg(B^3\gamma^j\gamma^j+B^2\gamma^j\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)+B^2\gamma^j\left(\frac{\zeta^i}{\gamma}-1\right)\\
&\qquad +B\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)+B^2\gamma^j\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\\
&\qquad +B\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)+\frac{B\gamma^j}{\gamma^j}\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(\frac{\zeta^i}{\gamma}-1\right)\\
&\qquad -\frac{7}{\gamma^j}\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\Bigg)\frac{1}{\zeta^i}\sum_{1\le k\le r}\big|\gamma^k\\
&\qquad -\zeta^i\big|\Bigg)\left(A\varphi o\acute{\upsilon}\ \left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\le\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\right)
\end{aligned}
$$

$$= \frac{1}{8} \sum_{1 \le i \le m_{12}} \sum_{1 \le j \le r} \left(\left(B^2 \gamma^j \left(B\gamma^j + \left(\frac{\zeta^i}{\gamma} - 1 \right) \right) \right. \right.$$
$$+ \left(\frac{\gamma^j}{2\beta} - \frac{\gamma^j}{2\alpha} + \frac{\gamma^j}{2\gamma} - \frac{1}{2} \right) \left(2B^2\gamma^j + 2B \left(\frac{\zeta^i}{\gamma} - 1 \right) + B \left(\frac{\gamma^j}{2\beta} - \frac{\gamma^j}{2\alpha} + \frac{\gamma^j}{2\gamma} - \frac{1}{2} \right) \right.$$
$$\left. \left. \left. - \frac{7}{\gamma^j} \left(\frac{\zeta^i}{\gamma} - 1 \right) \left(\frac{\gamma^j}{2\beta} - \frac{\gamma^j}{2\alpha} + \frac{\gamma^j}{2\gamma} - \frac{1}{2} \right) \right) \right) \frac{1}{\zeta^i} S(i) \right)$$

$$\ge \frac{1}{8} \sum_{1 \le i \le m_{12}} \sum_{1 \le j \le r} \left(\left(B^3 \gamma^j \gamma^j \right. \right.$$
$$+ \left(\frac{\gamma^j}{2\beta} - \frac{\gamma^j}{2\alpha} + \frac{\gamma^j}{2\gamma} - \frac{1}{2} \right) \left(2B^2\gamma^j + 2B \left(\frac{\zeta^i}{\gamma} - 1 \right) + B \left(\frac{\gamma^j}{2\beta} - \frac{\gamma^j}{2\alpha} + \frac{\gamma^j}{2\gamma} - \frac{1}{2} \right) \right.$$
$$\left. \left. \left. - \frac{7}{\gamma^j} \left(\frac{\zeta^i}{\gamma} - 1 \right) \left(\frac{\gamma^j}{2\beta} - \frac{\gamma^j}{2\alpha} + \frac{\gamma^j}{2\gamma} - \frac{1}{2} \right) \right) \right) \frac{1}{\zeta^i} S(i) \right)$$

23. <u>Απλοποίηση της Δ_{23}</u>

$$\Delta 23 = \sum_{1 \le k \le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1 \le i \le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1 \le j \le m_{12}} f^{r+m_1+m_2+m_3+j} \left(\frac{1}{\delta^i} - \frac{1}{\varepsilon^k} \right)$$
$$- \sum_{1 \le k \le m_{13}} p_3^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1 \le i \le m_{23}} p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1 \le j \le m_{12}} p_1^{r+m_1+m_2+m_3+j} \left(\frac{1}{\delta^i} - \frac{1}{\varepsilon^k} \right)$$

$$= \sum_{1 \le k \le m_{13}} \sum_{1 \le i \le m_{23}} \sum_{1 \le j \le m_{12}} \left(\frac{1}{2} \left(G\varepsilon^k + p_3^k \right) \frac{1}{2} \left(B\delta^i + p_2^i \right) \frac{1}{2} \left(B\zeta^j + p_1^j \right) - p_3^k p_2^i p_1^j \right) \left(\frac{1}{\delta^i} - \frac{1}{\varepsilon^k} \right)$$
$$= \frac{1}{8} \sum_{1 \le k \le m_{13}} \sum_{1 \le i \le m_{23}} \sum_{1 \le j \le m_{12}} \left(GB\delta^i \varepsilon^k \zeta^j + GB\delta^i \varepsilon^k p_1^j + GB\zeta^j \varepsilon^k p_2^i + G\varepsilon^k p_2^i p_1^j + B^2 \delta^i \zeta^j p_3^k + B\delta^i p_3^k p_1^j \right.$$
$$\left. + B\zeta^j p_3^k p_2^i - 7 p_3^k p_2^i p_1^j \right) \left(\frac{1}{\delta^i} - \frac{1}{\varepsilon^k} \right)$$

$$\geq \frac{1}{8}\sum_{1\leq k\leq m_{13}}\sum_{1\leq i\leq m_{23}}\sum_{1\leq j\leq m_{12}}\left(B^2\delta^i\varepsilon^k\zeta^j + B^2\delta^i\varepsilon^k p_1^j + B^2\zeta^j\varepsilon^k p_2^i + B\varepsilon^k p_2^i p_1^j + B^2\delta^i\zeta^j p_3^k + B\delta^i p_3^k p_1^j + B\zeta^j p_3^k p_2^i - 7p_3^k p_2^i p_1^j\right)\left(\frac{1}{\delta^i} - \frac{1}{\varepsilon^k}\right)(A\varphi o\acute{\upsilon}\ G \geq B)$$

$$= \frac{1}{8}\sum_{1\leq k\leq m_{13}}\sum_{1\leq i\leq m_{23}}\sum_{1\leq j\leq m_{12}}\left(B^2\delta^i\varepsilon^k\zeta^j + B^2\delta^i\varepsilon^k p_1^j + B^2\zeta^j\varepsilon^k p_2^i + B\varepsilon^k p_2^i p_1^j + B^2\delta^i\zeta^j p_3^k + B\delta^i p_3^k p_1^j + B\zeta^j p_3^k p_2^i - 7p_3^k p_2^i p_1^j\right)\left(\frac{1}{\delta^i} - \frac{1}{\varepsilon^k}\right)$$

$$= \frac{1}{8}\sum_{1\leq k\leq m_{13}}\sum_{1\leq i\leq m_{23}}\sum_{1\leq j\leq m_{12}}\left(B^2\delta^i\varepsilon^k\zeta^j + B^2\delta^i\varepsilon^k\left(\frac{\zeta^j}{\beta} - 1\right) + B^2\zeta^j\varepsilon^k\left(\frac{\delta^i}{\alpha} - 1\right) + B\varepsilon^k\left(\frac{\delta^i}{\alpha} - 1\right)\left(\frac{\zeta^j}{\beta} - 1\right) + B^2\delta^i\zeta^j\left(\frac{\varepsilon^k}{\gamma} - 1\right) + B\delta^i\left(\frac{\varepsilon^k}{\gamma} - 1\right)\left(\frac{\zeta^j}{\beta} - 1\right) + B\zeta^j\left(\frac{\varepsilon^k}{\gamma} - 1\right)\left(\frac{\delta^i}{\alpha} - 1\right) - 7\left(\frac{\varepsilon^k}{\gamma} - 1\right)\left(\frac{\delta^i}{\alpha} - 1\right)\left(\frac{\zeta^j}{\beta} - 1\right)\right)\left(\frac{1}{\delta^i} - \frac{1}{\varepsilon^k}\right)$$

24. Απλοποίηση της Δ_{24}

$$\Delta 24 = \sum_{1\leq k\leq m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k}\sum_{1\leq i\leq m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i}\sum_{1\leq j\leq m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\delta^i} - \frac{1}{\varepsilon^k}\right) - \sum_{1\leq k\leq m_{13}} p_3^{r+m_1+m_2+m_3+m_{12}+k}\sum_{1\leq i\leq m_{23}} p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i}\sum_{1\leq j\leq m_{13}} p_1^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\delta^i} - \frac{1}{\varepsilon^k}\right)$$

$$= \sum_{1\leq i\leq m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i}\sum_{1\leq j<k\leq m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\delta^i} - \frac{1}{\varepsilon^k}\right) - \sum_{1\leq i\leq m_{23}} p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i}\sum_{1\leq j<k\leq m_{13}} p_3^{r+m_1+m_2+m_3+m_{12}+k} p_1^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\delta^i} - \frac{1}{\varepsilon^k}\right)$$

$= \sum_{1\leq i\leq m_{23}}\sum_{1\leq j<k\leq m_{13}}\left(\frac{1}{8}\left(B\delta^i + p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i}\right)\left(A\varepsilon^j + p_3^{r+m_1+m_2+m_3+m_{12}+k}\right)\left(A\varepsilon^k + p_1^{r+m_1+m_2+m_3+m_{12}+j}\right) - p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} p_3^{r+m_1+m_2+m_3+m_{12}+k} p_1^{r+m_1+m_2+m_3+m_{12}+j}\right)\left(\frac{1}{\delta^i} - \frac{1}{\varepsilon^k}\right)$

$= \sum_{1\leq i\leq m_{23}}\sum_{1\leq j<k\leq m_{13}}\left(\frac{1}{8}\left(A^2B\delta^i\varepsilon^j\varepsilon^k + AB\delta^i\varepsilon^j p_1^j + A\varepsilon^k B\delta^i p_3^k + B\delta^i p_3^k p_1^j + A\varepsilon^k A\varepsilon^j p_2^i + A\varepsilon^j p_2^i p_1^j + A\varepsilon^k p_3^k p_2^i + p_3^k p_2^i p_1^j\right) - p_2^i p_3^k p_1^j\right)\left(\frac{1}{\delta^i} - \frac{1}{\varepsilon^k}\right)$

$$\geq \sum_{1\leq i\leq m_{23}} \sum_{1\leq j<k\leq m_{13}} \left(\frac{1}{8}\left(A^3\delta^i\varepsilon^j\varepsilon^k + A^2\delta^i\varepsilon^j p_1^j + A^2\delta^i\varepsilon^k p_3^k + A\delta^i p_3^k p_1^j + A^2\varepsilon^k\varepsilon^j p_2^i + A\varepsilon^j p_2^i p_1^j + A\varepsilon^k p_3^k p_2^i - 7p_3^k p_2^i p_1^j\right)\right)\left(\frac{1}{\delta^i} - \frac{1}{\varepsilon^k}\right) (\text{Αφού } B \geq A)$$

$$= \sum_{1\leq i\leq m_{23}} \sum_{1\leq j<k\leq m_{13}} \left(\frac{1}{8}\left(A^3\delta^i\varepsilon^j\varepsilon^k\left(\frac{1}{\delta^i} - \frac{1}{\varepsilon^k}\right) + A^2\delta^i\varepsilon^j\left(\frac{\varepsilon^j}{\alpha} - 1\right)\left(\frac{1}{\delta^i} - \frac{1}{\varepsilon^k}\right) + A^2\delta^i\varepsilon^k\left(\frac{\varepsilon^k}{\gamma} - 1\right)\left(\frac{1}{\delta^i} - \frac{1}{\varepsilon^k}\right) + A\delta^i\left(\frac{\varepsilon^k}{\gamma} - 1\right)\left(\frac{\varepsilon^j}{\alpha} - 1\right)\left(\frac{1}{\delta^i} - \frac{1}{\varepsilon^k}\right) + A^2\varepsilon^k\varepsilon^j\left(\frac{\delta^i}{\alpha} - 1\right)\left(\frac{1}{\delta^i} - \frac{1}{\varepsilon^k}\right) + A\varepsilon^j\left(\frac{\delta^i}{\alpha} - 1\right)\left(\frac{\varepsilon^j}{\alpha} - 1\right)\left(\frac{1}{\delta^i} - \frac{1}{\varepsilon^k}\right) + A\varepsilon^k\left(\frac{\varepsilon^k}{\gamma} - 1\right)\left(\frac{\delta^i}{\alpha} - 1\right)\left(\frac{1}{\delta^i} - \frac{1}{\varepsilon^k}\right) - 7\left(\frac{\varepsilon^k}{\gamma} - 1\right)\left(\frac{\delta^i}{\alpha} - 1\right)\left(\frac{\varepsilon^j}{\alpha} - 1\right)\left(\frac{1}{\delta^i} - \frac{1}{\varepsilon^k}\right)\right)\right)$$

$$\begin{aligned} = \sum_{1\leq i\leq m_{23}} \sum_{1\leq j<k\leq m_{13}} \Bigg(\frac{1}{8}\Bigg(& A^3\varepsilon^j\varepsilon^k\left(1 - \frac{\delta^i}{\varepsilon^k}\right) + A^2\varepsilon^j\left(\frac{\varepsilon^j}{\alpha} - 1\right)\left(1 - \frac{\delta^i}{\varepsilon^k}\right) + A^2\varepsilon^k\left(\frac{\varepsilon^k}{\gamma} - 1\right)\left(1 - \frac{\delta^i}{\varepsilon^k}\right) \\ & + A\left(\frac{\varepsilon^k}{\gamma} - 1\right)\left(\frac{\varepsilon^j}{\alpha} - 1\right)\left(1 - \frac{\delta^i}{\varepsilon^k}\right) + A^2\varepsilon^j\left(\frac{\delta^i}{\alpha} - 1\right)\left(\frac{\varepsilon^k}{\delta^i} - 1\right) \\ & + \frac{A\varepsilon^j}{\varepsilon^k}\left(\frac{\delta^i}{\alpha} - 1\right)\left(\frac{\varepsilon^j}{\alpha} - 1\right)\left(\frac{\varepsilon^k}{\delta^i} - 1\right) + A\left(\frac{\varepsilon^k}{\gamma} - 1\right)\left(\frac{\delta^i}{\alpha} - 1\right)\left(\frac{\varepsilon^k}{\delta^i} - 1\right) \\ & - \frac{7}{\varepsilon^k}\left(\frac{\varepsilon^k}{\gamma} - 1\right)\left(\frac{\delta^i}{\alpha} - 1\right)\left(\frac{\varepsilon^j}{\alpha} - 1\right)\left(\frac{\varepsilon^k}{\delta^i} - 1\right)\Bigg)\Bigg) \end{aligned}$$

$$\begin{aligned} = \frac{1}{8}\sum_{1\leq i\leq m_{23}} \sum_{1\leq j\leq m_{13}} \Bigg(\Bigg(& A^3\varepsilon^j\varepsilon^j + A^2\varepsilon^j\left(\frac{\varepsilon^j}{\alpha} - 1\right) + A^2\varepsilon^j\left(\frac{\varepsilon^j}{\gamma} - 1\right) + A\left(\frac{\varepsilon^j}{\gamma} - 1\right)\left(\frac{\varepsilon^j}{\alpha} - 1\right) \\ & + A^2\varepsilon^j\left(\frac{\delta^i}{\alpha} - 1\right) + \frac{A\varepsilon^j}{\varepsilon^j}\left(\frac{\delta^i}{\alpha} - 1\right)\left(\frac{\varepsilon^j}{\alpha} - 1\right) + A\left(\frac{\varepsilon^j}{\gamma} - 1\right)\left(\frac{\delta^i}{\alpha} - 1\right) \\ & - \frac{7}{\varepsilon^j}\left(\frac{\varepsilon^j}{\gamma} - 1\right)\left(\frac{\delta^i}{\alpha} - 1\right)\left(\frac{\varepsilon^j}{\alpha} - 1\right)\Bigg) \sum_{1\leq k\leq m_{13}} \left|\frac{\varepsilon^k}{\delta^i} - 1\right|\Bigg) \end{aligned}$$

$$\begin{aligned} = \frac{1}{8}\sum_{1\leq i\leq m_{23}} \sum_{1\leq j\leq m_{13}} \Bigg(\Bigg(& A^3\varepsilon^j\varepsilon^j + A^2\varepsilon^j\left(\frac{\varepsilon^j}{\alpha} - 1\right) + A^2\varepsilon^j\left(\frac{\varepsilon^j}{\gamma} - 1\right) + A\left(\frac{\varepsilon^j}{\gamma} - 1\right)\left(\frac{\varepsilon^j}{\alpha} - 1\right) \\ & + A^2\varepsilon^j\left(\frac{\delta^i}{\alpha} - 1\right) + \frac{A\varepsilon^j}{\varepsilon^j}\left(\frac{\delta^i}{\alpha} - 1\right)\left(\frac{\varepsilon^j}{\alpha} - 1\right) + A\left(\frac{\varepsilon^j}{\gamma} - 1\right)\left(\frac{\delta^i}{\alpha} - 1\right) \\ & - \frac{7}{\varepsilon^j}\left(\frac{\varepsilon^j}{\gamma} - 1\right)\left(\frac{\delta^i}{\alpha} - 1\right)\left(\frac{\varepsilon^j}{\alpha} - 1\right)\Bigg)\frac{1}{\delta^i} \sum_{1\leq k\leq m_{13}} \left|\varepsilon^k - \delta^i\right|\Bigg) \end{aligned}$$

$$\geq \frac{1}{8}\sum_{1\leq i\leq m_{23}}\sum_{1\leq j\leq m_{13}}\Bigg(\Bigg(A^3\varepsilon^j\varepsilon^j + A^2\varepsilon^j\left(\frac{\varepsilon^j}{\gamma}-1\right)+A^2\varepsilon^j\left(\frac{\varepsilon^j}{\gamma}-1\right)+A^2\varepsilon^j\left(\frac{\delta^i}{\alpha}-1\right)$$

$$+\frac{A\varepsilon^j}{\varepsilon^j}\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\varepsilon^j}{\gamma}-1\right)+A\left(\frac{\varepsilon^j}{\gamma}-1\right)\left(\frac{\delta^i}{\alpha}-1\right)$$

$$-\frac{7}{\varepsilon^j}\left(\frac{\varepsilon^j}{\gamma}-1\right)\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\varepsilon^j}{\alpha}-1\right)\Bigg)\frac{1}{\delta^i}S(i)\Bigg)(A\varphi o\acute{\upsilon}\ \gamma\geq\beta)$$

$$=\frac{1}{8}\sum\nolimits_{1\leq i\leq m_{23}}\sum\nolimits_{1\leq j\leq m_{13}}\Bigg(\frac{1}{\delta^i}\Bigg(A^3\varepsilon^j\varepsilon^j+\left(\frac{\varepsilon^j}{\gamma}-1\right)\Bigg(2A^2\varepsilon^j+2A\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\varepsilon^j}{\gamma}-1\right)+$$

$$A\left(\frac{\varepsilon^j}{\gamma}-1\right)\left(\frac{\varepsilon^j}{\gamma}-1\right)+A^2\varepsilon^j\left(\frac{\delta^i}{\alpha}-1\right)-\frac{7}{\varepsilon^j}\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\varepsilon^j}{\alpha}-1\right)\Bigg)\Bigg)S(i)\Bigg)$$

25. <u>Απλοποίηση της Δ_{25}</u>

$$\Delta 25=\sum_{1\leq k\leq m_{13}}f^{r+m_1+m_2+m_3+m_{12}+k}\sum_{1\leq i\leq m_{23}}f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i}\sum_{1\leq j\leq r}f^j\left(\frac{1}{\delta^i}-\frac{1}{\varepsilon^k}\right)$$

$$-\sum_{1\leq k\leq m_{13}}p_3^{r+m_1+m_2+m_3+m_{12}+k}\sum_{1\leq i\leq m_{23}}p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i}\sum_{1\leq j\leq r}p_1^j\left(\frac{1}{\delta^i}-\frac{1}{\varepsilon^k}\right)$$

$$=\sum_{1\leq k\leq m_{13}}\sum_{1\leq i\leq m_{23}}\sum_{1\leq j\leq r}\Big(f^{r+m_1+m_2+m_3+m_{12}+k}f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i}f^j$$

$$-p_3^{r+m_1+m_2+m_3+m_{12}+k}p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i}p_1^j\Big)\left(\frac{1}{\delta^i}-\frac{1}{\varepsilon^k}\right)$$

$$=\sum_{1\leq k\leq m_{13}}\sum_{1\leq i\leq m_{23}}\sum_{1\leq j\leq r}\Bigg(\frac{1}{2}\Big(G\varepsilon^k+p_3^{r+m_1+m_2+m_3+m_{12}+k}\Big)\frac{1}{2}\Big(B\delta^i+p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i}\Big)\frac{1}{2}\Bigg(\gamma^jB+\frac{\gamma^j}{\beta}$$

$$-1\Bigg)-p_3^{r+m_1+m_2+m_3+m_{12}+k}p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i}p_1^j\Bigg)\left(\frac{1}{\delta^i}-\frac{1}{\varepsilon^k}\right)$$

$$\geq\sum_{1\leq k\leq m_{13}}\sum_{1\leq i\leq m_{23}}\sum_{1\leq j\leq r}\Bigg(\frac{1}{2}\Big(G\varepsilon^k+p_3^{r+m_1+m_2+m_3+m_{12}+k}\Big)\frac{1}{2}\Big(B\delta^i+p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i}\Big)\frac{1}{2}\Big(\gamma^jB+p_1^j\Big)$$

$$-p_3^{r+m_1+m_2+m_3+m_{12}+k}p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i}p_1^j\Bigg)\left(\frac{1}{\delta^i}-\frac{1}{\varepsilon^k}\right)\left(A\varphi o\acute{\upsilon}\ \left(\frac{\gamma^j}{\beta}-1\right)\geq p_1^j\right)$$

$$=\sum_{1\leq k\leq m_{13}}\sum_{1\leq i\leq m_{23}}\sum_{1\leq j\leq r}\left(\frac{1}{4}\Big(G\varepsilon^kB\delta^i+G\varepsilon^kp_2^i+p_3^kB\delta^i+p_3^kp_2^i\Big)\frac{1}{2}\Big(\gamma^jB+p_1^j\Big)-p_3^kp_2^ip_1^j\right)\left(\frac{1}{\delta^i}-\frac{1}{\varepsilon^k}\right)$$

$$= \sum_{1\le k\le m_{13}} \sum_{1\le i\le m_{23}} \sum_{1\le j\le r} \Big(\frac{1}{8}\big(G\varepsilon^k B\delta^i\gamma^j B + G\varepsilon^k B\delta^i p_1^j + G\varepsilon^k p_2^i\gamma^j B + G\varepsilon^k p_2^i p_1^j + p_3^k B\delta^i\gamma^j B + p_3^k B\delta^i p_1^j + \gamma^j B p_3^k p_2^i + p_3^k p_2^i p_1^j\big) - p_3^k p_2^i p_1^j\Big)\left(\frac{1}{\delta^i} - \frac{1}{\varepsilon^k}\right)$$

$= \frac{1}{8}\sum_{1\le k\le m_{13}}\sum_{1\le i\le m_{23}}\sum_{1\le j\le r}\big(B^3\varepsilon^k\delta^i\gamma^j + B^2\varepsilon^k\delta^i p_1^j + B^2\varepsilon^k\gamma^j p_2^i + B\varepsilon^k p_2^i p_1^j + B^2 p_3^k\delta^i\gamma^j + p_3^k B\delta^i p_1^j + \gamma^j B p_3^k p_2^i - 7p_3^k p_2^i p_1^j\big)\left(\frac{1}{\delta^i} - \frac{1}{\varepsilon^k}\right)$ $(A\varphi o\acute{\upsilon}\ G \ge B)$

26. <u>Απλοποίηση της Δ_{26}</u>

$$\Delta 26 = \sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j}\left(\frac{1}{\delta^i} - \frac{1}{\delta^k}\right) - \sum_{1\le k\le m_{23}} p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_{23}} p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_{12}} p_1^{r+m_1+m_2+m_3+j}\left(\frac{1}{\delta^i} - \frac{1}{\delta^k}\right)$$

$$= \sum_{1\le k<i\le m_{23}} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} f^{r+m_1+m_2+m_3+j}\left(\frac{1}{\delta^i} - \frac{1}{\delta^k}\right) - \sum_{1\le k<i\le m_{23}} \sum_{1\le j\le m_{12}} p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} p_1^{r+m_1+m_2+m_3+j}\left(\frac{1}{\delta^i} - \frac{1}{\delta^k}\right)$$

$= \sum_{1\le k<i\le m_{23}}\sum_{1\le j\le m_{12}}\left(\frac{1}{2}\big(B\delta^k + p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k}\big)\frac{1}{2}\big(B\delta^i + p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i}\big)\frac{1}{2}\big(B\zeta^j + p_1^{r+m_1+m_2+m_3+j}\big) - p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} p_1^{r+m_1+m_2+m_3+j}\right)\left(\frac{1}{\delta^i} - \frac{1}{\delta^k}\right)$

$$= \sum_{1\le k<i\le m_{23}} \sum_{1\le j\le m_{12}} \frac{1}{8}\Big(\big(B^3\delta^k\delta^i\zeta^j + B^2\delta^k\delta^i p_1^j + B^2\delta^k\delta^i\zeta^j p_2^i + B\delta^k\zeta^j p_2^i p_1^j + B^2\delta^i p_3^k\zeta^j + B\delta^i p_3^k p_1^j + B\zeta^j p_3^k p_1^j + p_3^k p_2^i p_1^j\big) - p_3^k p_2^i p_1^j\Big)\left(\frac{1}{\delta^i} - \frac{1}{\delta^k}\right)$$

$$= \sum_{1\le k<i\le m_{23}} \sum_{1\le j\le m_{12}} \frac{1}{8}\Big(\Big(B^2\delta^k\delta^i\big(B\zeta^j + p_1^j + \zeta^j p_2^i\big) + B\delta^k\zeta^j p_2^i p_1^j + B^2\delta^i p_3^k \zeta^j + B\delta^i p_3^k p_1^j + B\zeta^j p_3^k p_1^j$$
$$- 7p_3^k p_2^i p_1^j\Big)\Big)\left(\frac{1}{\delta^i} - \frac{1}{\delta^k}\right)$$

$$= \sum_{1\le k<i\le m_{23}} \sum_{1\le j\le m_{12}} \left(\frac{1}{2}\left(\delta^k B + \frac{\delta^k}{\beta} - 1\right)\frac{1}{2}\left(B\delta^i + \frac{\delta^i}{\alpha} - 1\right)\frac{1}{2}\left(B\zeta^j + \frac{\zeta^j}{\beta} - 1\right)\right.$$
$$\left. - \left(\frac{\delta^k}{\beta} - 1\right)\left(\frac{\delta^i}{\alpha} - 1\right)\left(\frac{\zeta^j}{\beta} - 1\right)\right)\left(\frac{1}{\delta^i} - \frac{1}{\delta^k}\right)$$

$$= \frac{1}{8}\sum_{1\le k<i\le m_{23}} \sum_{1\le j\le m_{12}} \left(\left(B^3\delta^k\zeta^j\left(1 - \frac{\delta^i}{\delta^k}\right) + B^2\delta^k\left(\frac{\zeta^j}{\beta} - 1\right)\left(1 - \frac{\delta^i}{\delta^k}\right)\right.\right.$$
$$+ B^2\delta^k\zeta^j\left(\frac{\delta^i}{\alpha} - 1\right)\left(1 - \frac{\delta^i}{\delta^k}\right) + B\zeta^j\left(\frac{\delta^i}{\alpha} - 1\right)\left(\frac{\zeta^j}{\beta} - 1\right)\left(\frac{\delta^k}{\delta^i} - 1\right)$$
$$+ B^2\zeta^j\left(\frac{\delta^k}{\beta} - 1\right)\left(1 - \frac{\delta^i}{\delta^k}\right) + B\left(\frac{\delta^k}{\beta} - 1\right)\left(\frac{\zeta^j}{\beta} - 1\right)\left(1 - \frac{\delta^i}{\delta^k}\right)$$
$$+ \frac{B\zeta^j}{\delta^k}\left(\frac{\delta^k}{\beta} - 1\right)\left(\frac{\zeta^j}{\beta} - 1\right)\left(\frac{\delta^k}{\delta^i} - 1\right)$$
$$\left.\left. - \frac{7}{\delta^k}\left(\frac{\delta^k}{\beta} - 1\right)\left(\frac{\delta^i}{\alpha} - 1\right)\left(\frac{\zeta^j}{\beta} - 1\right)\left(\frac{\delta^k}{\delta^i} - 1\right)\right)\right)$$

$$= \frac{1}{8}\sum_{1\le k\le m_{23}} \sum_{1\le j\le m_{12}} \left(\left(B^3\delta^k\zeta^j + B^2\delta^k\left(\frac{\zeta^j}{\beta} - 1\right) + B^2\delta^k\zeta^j\left(\frac{\delta^i}{\alpha} - 1\right)\right.\right.$$
$$+ B\zeta^j\left(\frac{\delta^i}{\alpha} - 1\right)\left(\frac{\zeta^j}{\beta} - 1\right) + B^2\zeta^j\left(\frac{\delta^k}{\beta} - 1\right) + B\left(\frac{\delta^k}{\beta} - 1\right)\left(\frac{\zeta^j}{\beta} - 1\right)$$
$$+ \frac{B\zeta^j}{\delta^k}\left(\frac{\delta^k}{\beta} - 1\right)\left(\frac{\zeta^j}{\beta} - 1\right)$$
$$\left. - \frac{7}{\delta^k}\left(\frac{\delta^k}{\beta} - 1\right)\left(\frac{\delta^i}{\alpha} - 1\right)\left(\frac{\zeta^j}{\beta} - 1\right)\right)\frac{1}{\delta^k}\sum_{1\le i\le m_{23}}\left(\delta^k - \delta^i\right)\Bigg)$$

$$=\frac{1}{8}\sum_{1\le k\le m_{23}}\sum_{1\le j\le m_{12}}\left(\left(B^3\delta^k\zeta^j+B^2\delta^k\left(\frac{\zeta^j}{\beta}-1\right)+B^2\delta^k\zeta^j\left(\frac{\delta^k}{\alpha}-1\right)\right.\right.$$
$$+B\zeta^j\left(\frac{\delta^k}{\alpha}-1\right)\left(\frac{\zeta^j}{\beta}-1\right)+B^2\zeta^j\left(\frac{\delta^k}{\beta}-1\right)+B\left(\frac{\delta^k}{\beta}-1\right)\left(\frac{\zeta^j}{\beta}-1\right)$$
$$\left.\left.+\frac{B\zeta^j}{\delta^k}\left(\frac{\delta^k}{\beta}-1\right)\left(\frac{\zeta^j}{\beta}-1\right)-\frac{7}{\delta^k}\left(\frac{\delta^k}{\beta}-1\right)\left(\frac{\delta^k}{\alpha}-1\right)\left(\frac{\zeta^j}{\beta}-1\right)\right)\frac{1}{\delta^k}S(k)\right)$$

$$=\frac{1}{8}\sum_{1\le k\le m_{23}}\sum_{1\le j\le m_{12}}\left(\left(B^3\delta^k\zeta^j+B^2\delta^k\left(\frac{\zeta^j}{\beta}-1\right)+B^2\delta^k\zeta^j\left(\frac{\delta^k}{\alpha}-1\right)\right.\right.$$
$$+B\zeta^j\left(\frac{\delta^k}{\alpha}-1\right)\left(\frac{\zeta^j}{\beta}-1\right)+B^2\zeta^j\left(\frac{\delta^k}{\beta}-1\right)+B\left(\frac{\delta^k}{\beta}-1\right)\left(\frac{\zeta^j}{\beta}-1\right)$$
$$\left.\left.+\frac{B\zeta^j}{\delta^k}\left(\frac{\delta^k}{\beta}-1\right)\left(\frac{\zeta^j}{\beta}-1\right)-\frac{7}{\delta^k}\left(\frac{\delta^k}{\beta}-1\right)\left(\frac{\delta^k}{\alpha}-1\right)\left(\frac{\zeta^j}{\beta}-1\right)\right)\frac{1}{\delta^k}S(k)\right)$$

27. <u>Απλοποίηση της Δ_{27}</u>

$$\Delta 27=\sum_{1\le k\le m_{23}}f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k}\sum_{1\le i\le m_{23}}f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i}\sum_{1\le j\le m_{13}}f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\delta^i}-\frac{1}{\delta^k}\right)$$
$$-\sum_{1\le k\le m_{23}}p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k}\sum_{1\le i\le m_{23}}p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i}\sum_{1\le j\le m_{13}}p_1^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\delta^i}-\frac{1}{\delta^k}\right)$$

$$=\sum_{1\le k\le m_{23}}f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k}\sum_{1\le i\le m_{23}}f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i}\sum_{1\le j\le m_{13}}f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\delta^i}-\frac{1}{\delta^k}\right)$$
$$-\sum_{1\le k\le m_{23}}p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k}\sum_{1\le i\le m_{23}}p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i}\sum_{1\le j\le m_{13}}p_1^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\delta^i}-\frac{1}{\delta^k}\right)$$

$$= \sum_{1\le k<i\le m_{23}} \sum_{1\le j\le m_{13}} \left(\frac{1}{2}\left(\delta^k B+\frac{\delta^k}{\beta}-1\right)\frac{1}{2}\left(B\delta^i+\frac{\delta^i}{\alpha}-1\right)\frac{1}{2}\left(A\varepsilon^j+\frac{\varepsilon^j}{a}-1\right)\right)\left(\frac{1}{\delta^i}-\frac{1}{\delta^k}\right)$$

$$- \sum_{1\le k<i\le m_{23}} \sum_{1\le j\le m_{13}} \left(\left(\frac{\delta^k}{\beta}-1\right)\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\varepsilon^j}{\alpha}-1\right)\right)\left(\frac{1}{\delta^i}-\frac{1}{\delta^k}\right)$$

$$= \sum_{1\le k<i\le m_{23}} \sum_{1\le j\le m_{13}} \left(\frac{1}{8}\left(B\delta^k+p_3^k\right)\left(B\delta^i+p_2^i\right)\left(A\varepsilon^j+p_1^j\right)-p_3^k p_2^i p_1^j\right)\left(\frac{1}{\delta^i}-\frac{1}{\delta^k}\right)$$

$$= \sum_{1\le k<i\le m_{23}} \sum_{1\le j\le m_{13}} \left(\frac{1}{8}\left(B\delta^k B\delta^i+B\delta^k p_2^i+B\delta^i p_3^k+p_3^k p_2^i\right)\left(A\varepsilon^j+p_1^j\right)-p_3^k p_2^i p_1^j\right)\left(\frac{1}{\delta^i}-\frac{1}{\delta^k}\right)$$

$$= \sum_{1\le k<i\le m_{23}} \sum_{1\le j\le m_{13}} \left(\frac{1}{8}\left(B\delta^k B\delta^i A\varepsilon^j+B\delta^k B\delta^i p_1^j+B\delta^k p_2^i A\varepsilon^j+B\delta^k p_2^i p_1^j+B\delta^i p_3^k A\varepsilon^j\right.\right.$$
$$\left.\left.+B\delta^i p_3^k p_1^j+A\varepsilon^j p_3^k p_2^i+p_3^k p_2^i p_1^j\right)-p_3^k p_2^i p_1^j\right)\left(\frac{1}{\delta^i}-\frac{1}{\delta^k}\right)$$

$$= \frac{1}{8}\sum_{1\le k<i\le m_{23}} \sum_{1\le j\le m_{13}} \left(B\delta^k B\delta^i A\varepsilon^j+B\delta^k B\delta^i p_1^j+B\delta^k p_2^i A\varepsilon^j+B\delta^k p_2^i p_1^j+B\delta^i p_3^k A\varepsilon^j+B\delta^i p_3^k p_1^j\right.$$
$$\left.+A\varepsilon^j p_3^k p_2^i-7p_3^k p_2^i p_1^j\right)\left(\frac{1}{\delta^i}-\frac{1}{\delta^k}\right)$$

$$= \frac{1}{8}\sum_{1\le k<i\le m_{23}} \sum_{1\le j\le m_{13}} \left(B\delta^k B\delta^i A\varepsilon^j+B\delta^k B\delta^i p_1^j+B\delta^k p_2^i A\varepsilon^j+B\delta^k p_2^i p_1^j+B\delta^i p_3^k A\varepsilon^j+B\delta^i p_3^k p_1^j\right.$$
$$\left.+A\varepsilon^j p_3^k p_2^i-7p_3^k p_2^i p_1^j\right)\left(\frac{1}{\delta^i}-\frac{1}{\delta^k}\right)$$

$$\ge \frac{1}{8}\sum_{1\le k<i\le m_{23}} \sum_{1\le j\le m_{13}} \left(A^3\delta^k\delta^i\varepsilon^j+A^2\delta^k\delta^i p_1^j+A^2\delta^k p_2^i\varepsilon^j+A\delta^k p_2^i p_1^j+A^2\delta^i p_3^k\varepsilon^j+A\delta^i p_3^k p_1^j\right.$$
$$\left.+A\varepsilon^j p_3^k p_2^i-7p_3^k p_2^i p_1^j\right)\left(\frac{1}{\delta^i}-\frac{1}{\delta^k}\right)(A\varphi o\acute{\upsilon}\ B\ge A)$$

$$= \frac{1}{8} \sum_{1 \le k < i \le m_{23}} \sum_{1 \le j \le m_{13}} \left(A^3 \delta^k \delta^i \varepsilon^j + A^2 \delta^k \delta^i \left(\frac{\varepsilon^j}{\alpha} - 1 \right) + A^2 \delta^k \left(\frac{\delta^i}{\alpha} - 1 \right) \varepsilon^j + A \delta^k \left(\frac{\delta^i}{\alpha} - 1 \right) \left(\frac{\varepsilon^j}{\alpha} - 1 \right) \right.$$
$$+ A^2 \delta^i \left(\frac{\delta^k}{\beta} - 1 \right) \varepsilon^j + A \delta^i \left(\frac{\delta^k}{\beta} - 1 \right) \left(\frac{\varepsilon^j}{\alpha} - 1 \right) + A \varepsilon^j \left(\frac{\delta^k}{\beta} - 1 \right) \left(\frac{\delta^i}{\alpha} - 1 \right)$$
$$\left. - 7 \left(\frac{\delta^k}{\beta} - 1 \right) \left(\frac{\delta^i}{\alpha} - 1 \right) \left(\frac{\varepsilon^j}{\alpha} - 1 \right) \right) \left(\frac{1}{\delta^i} - \frac{1}{\delta^k} \right)$$

$$= \frac{1}{8} \sum_{1 \le k < i \le m_{23}} \sum_{1 \le j \le m_{13}} \left(A^3 \varepsilon^j \delta^k \left(1 - \frac{\delta^i}{\delta^k} \right) + A^2 \delta^k \left(\frac{\varepsilon^j}{\alpha} - 1 \right) \left(1 - \frac{\delta^i}{\delta^k} \right) + A^2 \varepsilon^j \left(\frac{\delta^i}{\alpha} - 1 \right) \left(\frac{\delta^k}{\delta^i} - 1 \right) \right.$$
$$+ A \left(\frac{\delta^i}{\alpha} - 1 \right) \left(\frac{\varepsilon^j}{\alpha} - 1 \right) \left(\frac{\delta^k}{\delta^i} - 1 \right) + A^2 \varepsilon^j \left(\frac{\delta^k}{\beta} - 1 \right) \left(1 - \frac{\delta^i}{\delta^k} \right)$$
$$+ A \left(\frac{\delta^k}{\beta} - 1 \right) \left(\frac{\varepsilon^j}{\alpha} - 1 \right) \left(1 - \frac{\delta^i}{\delta^k} \right) + \frac{A \varepsilon^j}{\delta^k} \left(\frac{\delta^k}{\beta} - 1 \right) \left(\frac{\delta^i}{\alpha} - 1 \right) \left(\frac{\delta^k}{\delta^i} - 1 \right)$$
$$\left. - \frac{7}{\delta^k} \left(\frac{\delta^k}{\beta} - 1 \right) \left(\frac{\delta^i}{\alpha} - 1 \right) \left(\frac{\varepsilon^j}{\alpha} - 1 \right) \left(\frac{\delta^k}{\delta^i} - 1 \right) \right)$$

$$= \frac{1}{8} \sum_{1 \le k \le m_{23}} \sum_{1 \le j \le m_{13}} \left(\left(A^3 \varepsilon^j \delta^k + A^2 \delta^k \left(\frac{\varepsilon^j}{\alpha} - 1 \right) + A^2 \varepsilon^j \left(\frac{\delta^k}{\alpha} - 1 \right) + A \left(\frac{\delta^k}{\alpha} - 1 \right) \left(\frac{\varepsilon^j}{\alpha} - 1 \right) \right. \right.$$
$$+ A^2 \varepsilon^j \left(\frac{\delta^k}{\beta} - 1 \right) + A \left(\frac{\delta^k}{\beta} - 1 \right) \left(\frac{\varepsilon^j}{\alpha} - 1 \right) + \frac{A \varepsilon^j}{\delta^k} \left(\frac{\delta^k}{\beta} - 1 \right) \left(\frac{\delta^k}{\alpha} - 1 \right)$$
$$\left. \left. - \frac{7}{\delta^k} \left(\frac{\delta^k}{\beta} - 1 \right) \left(\frac{\delta^k}{\alpha} - 1 \right) \left(\frac{\varepsilon^j}{\alpha} - 1 \right) \right) \sum_{1 \le i \le m_{23}} \left(1 - \frac{\delta^i}{\delta^k} \right) \right)$$

$$\ge \frac{1}{8} \sum_{1 \le k \le m_{23}} \sum_{1 \le j \le m_{13}} \left(\left(A^3 \varepsilon^j \delta^k + A^2 \delta^k \left(\frac{\varepsilon^j}{\alpha} - 1 \right) + A^2 \varepsilon^j \left(\frac{\delta^k}{\beta} - 1 \right) + A \left(\frac{\delta^k}{\beta} - 1 \right) \left(\frac{\varepsilon^j}{\alpha} - 1 \right) \right. \right.$$
$$+ A^2 \varepsilon^j \left(\frac{\delta^k}{\beta} - 1 \right) + A \left(\frac{\delta^k}{\beta} - 1 \right) \left(\frac{\varepsilon^j}{\alpha} - 1 \right) + \frac{A \varepsilon^j}{\delta^k} \left(\frac{\delta^k}{\beta} - 1 \right) \left(\frac{\delta^k}{\beta} - 1 \right)$$
$$\left. \left. - \frac{7}{\delta^k} \left(\frac{\delta^k}{\beta} - 1 \right) \left(\frac{\delta^k}{\alpha} - 1 \right) \left(\frac{\varepsilon^j}{\alpha} - 1 \right) \right) \frac{1}{\delta^k} \sum_{1 \le i \le m_{23}} \left(\delta^k - \delta^i \right) \right) (Αφού\ \beta \ge \alpha)$$

$$\ge \frac{1}{8} \sum_{1 \le k \le m_{23}} \sum_{1 \le j \le m_{13}} \left(\left(A^3 \varepsilon^j \delta^k + \left(\frac{\delta^k}{\beta} - 1 \right) \left(2A^2 \varepsilon^j + 2A \left(\frac{\varepsilon^j}{\alpha} - 1 \right) + A \left(\frac{\delta^k}{\beta} - 1 \right) - \frac{7}{\delta^k} \left(\frac{\delta^k}{\alpha} - 1 \right) \left(\frac{\varepsilon^j}{\alpha} - 1 \right) \right) \right) \frac{1}{\delta^k} S(k) \right)$$

28. <u>Απλοποίηση της Δ_{28}</u>

$$\Delta 28 = \sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le r} f^j \left(\frac{1}{\delta^i}-\frac{1}{\delta^k}\right)$$
$$- \sum_{1\le k\le m_{23}} p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_{23}} p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le r} p_1^j \left(\frac{1}{\delta^i}-\frac{1}{\delta^k}\right)$$

$$= \sum_{1\le k<i\le m_{23}} \sum_{1\le j\le r} \left(f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} f^j\right.$$
$$\left. - p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} p_1^j\right) \left(\frac{1}{\delta^i}-\frac{1}{\delta^k}\right)$$

$$= \sum_{1\le k<i\le m_{23}} \sum_{1\le j\le r} \left(\frac{1}{2}\left(B\delta^k + p_3^k\right)\frac{1}{2}\left(B\delta^i + p_2^i\right)\frac{1}{2}\left(\gamma^j B + \frac{\gamma^j}{\beta} - 1\right) - p_3^k p_2^i p_1^j\right)\left(\frac{1}{\delta^i}-\frac{1}{\delta^k}\right)$$
$$\ge \sum_{1\le k<i\le m_{23}} \sum_{1\le j\le r} \left(\frac{1}{8}\left(B\delta^k B\delta^i + B\delta^k p_2^i + B\delta^i p_3^k + p_3^k p_2^i\right)\left(\gamma^j B + p_1^j\right) - p_3^k p_2^i p_1^j\right)\left(\frac{1}{\delta^i}\right.$$
$$\left.-\frac{1}{\delta^k}\right)\left(Αφού\ \frac{\gamma^j}{\beta} - 1 \ge \frac{\gamma^j}{2\beta} - \frac{\gamma^j}{2\alpha} + \frac{\gamma^j}{2\gamma} - \frac{1}{2}\left(= p_1^j\right)\right)$$
$$= \sum_{1\le k<i\le m_{23}} \sum_{1\le j\le r} \left(\frac{1}{8}\left(B\delta^k B\delta^i + B\delta^k p_2^i + B\delta^i p_3^k + p_3^k p_2^i\right)\left(\gamma^j B + p_1^j\right) - p_3^k p_2^i p_1^j\right)\left(\frac{1}{\delta^i}-\frac{1}{\delta^k}\right)$$
$$= \sum_{1\le k<i\le m_{23}} \sum_{1\le j\le r} \left(\frac{1}{8}\left(B^3\delta^k\delta^i\gamma^j + B^2\delta^k\delta^i p_1^j + B^2\delta^k\gamma^j p_2^i + B\delta^k p_2^i p_1^j + B^2\delta^i p_3^k\gamma^j + B\delta^i p_3^k p_1^j\right.\right.$$
$$\left.\left. + Bp_3^k p_2^i \gamma^j - 7p_3^k p_2^i p_1^j\right)\right)\left(\frac{1}{\delta^i}-\frac{1}{\delta^k}\right)$$

$$= \sum_{1\le k<i\le m_{23}} \sum_{1\le j\le r} \left(\frac{1}{8}\left(B^3\delta^k\delta^i\gamma^j + B^2\delta^k\delta^i\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right) + B^2\delta^k\gamma^j\left(\frac{\delta^i}{\alpha}-1\right)\right.\right.$$
$$+ B\delta^k\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right) + B^2\delta^i\gamma^j\left(\frac{\delta^k}{\beta}-1\right)$$
$$+ B\delta^i\left(\frac{\delta^k}{\beta}-1\right)\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right) + B\gamma^j\left(\frac{\delta^k}{\beta}-1\right)\left(\frac{\delta^i}{\alpha}-1\right)$$
$$\left.\left. - 7\left(\frac{\delta^k}{\beta}-1\right)\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\right)\right)\left(\frac{1}{\delta^i}-\frac{1}{\delta^k}\right)$$

$$= \sum_{1\le k<i\le m_{23}} \sum_{1\le j\le r} \left(\frac{1}{8}\left(B^3\delta^k\gamma^j\left(1-\frac{\delta^i}{\delta^k}\right) + B^2\delta^k\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(1-\frac{\delta^i}{\delta^k}\right)\right.\right.$$
$$+ B^2\gamma^j\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\delta^k}{\delta^i}-1\right) + B\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(\frac{\delta^k}{\delta^i}-1\right)$$
$$+ B^2\gamma^j\left(\frac{\delta^k}{\beta}-1\right)\left(1-\frac{\delta^i}{\delta^k}\right) + B\left(\frac{\delta^k}{\beta}-1\right)\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(1-\frac{\delta^i}{\delta^k}\right)$$
$$+ \frac{B\gamma^j}{\delta^k}\left(\frac{\delta^k}{\beta}-1\right)\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\delta^k}{\delta^i}-1\right)$$
$$\left.\left. - \frac{7}{\delta^k}\left(\frac{\delta^k}{\beta}-1\right)\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(\frac{\delta^k}{\delta^i}-1\right)\right)\right)$$

$$= \frac{1}{8}\sum_{1\le k\le m_{23}} \sum_{1\le j\le r} \left(\left(B^3\delta^k\gamma^j + B^2\delta^k\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right) + B^2\gamma^j\left(\frac{\delta^k}{\alpha}-1\right)\right.\right.$$
$$+ B\left(\frac{\delta^k}{\alpha}-1\right)\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right) + B^2\gamma^j\left(\frac{\delta^k}{\beta}-1\right)$$
$$+ B\left(\frac{\delta^k}{\beta}-1\right)\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right) + \frac{B\gamma^j}{\delta^k}\left(\frac{\delta^k}{\beta}-1\right)\left(\frac{\delta^k}{\alpha}-1\right)$$
$$\left.\left. - \frac{7}{\delta^k}\left(\frac{\delta^k}{\beta}-1\right)\left(\frac{\delta^k}{\alpha}-1\right)\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\right)\sum_{1\le i\le m_{23}}\left(1-\frac{\delta^i}{\delta^k}\right)\right)$$

$$= \frac{1}{8}\sum_{1\le k\le m_{23}}\sum_{1\le j\le r}\Bigg(\Bigg(B^3\delta^k\gamma^j + B^2\delta^k\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)+B^2\gamma^j\left(\frac{\delta^k}{\alpha}-1\right)$$

$$+B\left(\frac{\delta^k}{\alpha}-1\right)\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)+B^2\gamma^j\left(\frac{\delta^k}{\beta}-1\right)$$

$$+B\left(\frac{\delta^k}{\beta}-1\right)\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)+\frac{B\gamma^j}{\delta^k}\left(\frac{\delta^k}{\beta}-1\right)\left(\frac{\delta^k}{\alpha}-1\right)$$

$$-\frac{7}{\delta^k}\left(\frac{\delta^k}{\beta}-1\right)\left(\frac{\delta^k}{\alpha}-1\right)\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\Bigg)\frac{1}{\delta^k}\sum_{1\le i\le m_{23}}\left(\delta^k-\delta^i\right)\Bigg)$$

$$\ge \frac{1}{8}\sum_{1\le k\le m_{23}}\sum_{1\le j\le r}\Bigg(\Bigg(B^3\delta^k\gamma^j$$

$$+\left(\frac{\delta^k}{\beta}-1\right)\Bigg(2B^2\gamma^j+2B\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)+B\left(\frac{\delta^k}{\beta}-1\right)$$

$$-\frac{7}{\delta^k}\left(\frac{\delta^k}{\alpha}-1\right)\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\Bigg)\Bigg)\frac{1}{\delta^k}S(k)\Bigg)\ (Αφού\ \beta \ge \alpha)$$

29. <u>Απλοποίηση της $\varDelta_{29}$</u>

$$\varDelta_{29} = \sum_{1\le k\le r}\sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i}\sum_{1\le j\le r}\left(\frac{1}{\delta^i}-\frac{1}{\gamma^k}\right)$$

$$-\sum_{1\le k\le r} p_3^k \sum_{1\le i\le m_{23}} p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i}\sum_{1\le j\le r} p_1^j\left(\frac{1}{\delta^i}-\frac{1}{\gamma^k}\right)$$

$$= \sum_{1\le i\le m_{23}}\sum_{1\le k<j\le r}\left(f^k f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} f^j - p_3^k p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} p_1^j\right)\left(\frac{1}{\delta^i}-\frac{1}{\gamma^k}\right)$$

$$= \sum_{1\le i\le m_{23}} \sum_{1\le k<j\le r} \left(\frac{1}{2}\left(\gamma^k B + \frac{\gamma^k}{\beta} - 1\right)\frac{1}{2}\left(B\delta^i + p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i}\right)\frac{1}{2}\left(\gamma^j B + \frac{\gamma^j}{\beta} - 1\right)\right.$$

$$\left. - p_3^k p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} p_1^j\right)\left(\frac{1}{\delta^i} - \frac{1}{\gamma^k}\right)$$

$$\ge \sum_{1\le i\le m_{23}} \sum_{1\le k<j\le r} \left(\frac{1}{8}\left(\gamma^k B + p_3^k\right)\left(B\delta^i + p_2^i\right)\left(\gamma^j B + p_1^j\right)\right.$$

$$\left. - p_3^k p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} p_1^j\right)\left(\frac{1}{\delta^i} - \frac{1}{\gamma^k}\right)$$

$$= \sum_{1\le i\le m_{23}} \sum_{1\le k<j\le r} \left(\frac{1}{8}\left(\gamma^k B + p_3^k\right)\left(B\delta^i + p_2^i\right)\left(\gamma^j B + p_1^j\right)\right.$$

$$\left. - p_3^k p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} p_1^j\right)\left(\frac{1}{\delta^i} - \frac{1}{\gamma^k}\right)$$

$$= \sum_{1\le i\le m_{23}} \sum_{1\le k<j\le r} \left(\frac{1}{8}\left(\gamma^k BB\delta^i + \gamma^k B p_2^i + B\delta^i p_3^k + p_3^k p_2^i\right)\left(\gamma^j B + p_1^j\right)\right.$$

$$\left. - p_3^k p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} p_1^j\right)\left(\frac{1}{\delta^i} - \frac{1}{\gamma^k}\right)$$

$$= \sum_{1\le i\le m_{23}} \sum_{1\le k<j\le r} \left(\frac{1}{8}\left(\gamma^k\gamma^j BBB\delta^i + \gamma^k BB\delta^i p_1^j + \gamma^k\gamma^j BB p_2^i + \gamma^k B p_2^i p_1^j\right.\right.$$

$$\left.\left. + B\delta^i p_3^k \gamma^j B + B\delta^i p_3^k p_1^j + p_3^k p_2^i \gamma^j B - 7 p_3^k p_2^i p_1^j\right)\right)\left(\frac{1}{\delta^i} - \frac{1}{\gamma^k}\right)$$

$$= \sum_{1\le i\le m_{23}} \sum_{1\le k<j\le r} \left(\frac{1}{8}\left(B^3\gamma^k\gamma^j\delta^i + B^2\gamma^k\delta^i\left(\frac{\gamma^j}{2\beta} + \frac{\gamma^j}{2\alpha} - \frac{\gamma^j}{2\gamma} - \frac{1}{2}\right) + B^2\gamma^k\gamma^j\left(\frac{\delta^i}{\alpha} - 1\right)\right.\right.$$

$$+ B\gamma^k\left(\frac{\delta^i}{\alpha} - 1\right)\left(\frac{\gamma^j}{2\beta} + \frac{\gamma^j}{2\alpha} - \frac{\gamma^j}{2\gamma} - \frac{1}{2}\right) + B\delta^i\left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2}\right)\gamma^j B$$

$$+ B\delta^i\left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2}\right)\left(\frac{\gamma^j}{2\beta} + \frac{\gamma^j}{2\alpha} - \frac{\gamma^j}{2\gamma} - \frac{1}{2}\right)$$

$$+ \left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2}\right)\left(\frac{\delta^i}{\alpha} - 1\right)\gamma^j B$$

$$\left.\left. - 7\left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2}\right)\left(\frac{\delta^i}{\alpha} - 1\right)\left(\frac{\gamma^j}{2\beta} + \frac{\gamma^j}{2\alpha} - \frac{\gamma^j}{2\gamma} - \frac{1}{2}\right)\right)\right)\left(\frac{1}{\delta^i} - \frac{1}{\gamma^k}\right)$$

$$
\begin{aligned}
=\frac{1}{8}\sum_{1\le i\le m_{23}}\sum_{1\le j\le r}\Bigg(\Bigg(&B^3\gamma^k\gamma^j+B^2\gamma^k\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)+B^2\gamma^j\left(\frac{\delta^i}{\alpha}-1\right)\\
&+B\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)+B^2\gamma^j\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\\
&+B\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\\
&+B\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(\frac{\delta^i}{\alpha}-1\right)\\
&-\frac{7}{\delta^i}\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(\frac{\delta^i}{\alpha}-1\right)\Bigg)\sum_{1\le k\le r}\Bigg(\frac{\gamma^k}{\delta^i}\\
&-1\Bigg)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
\ge\frac{1}{8}\sum_{1\le i\le m_{23}}\sum_{1\le j\le r}\Bigg(\Bigg(&B^3\gamma^j\gamma^j+B^2\gamma^j\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)+B^2\gamma^j\left(\frac{\delta^i}{\alpha}-1\right)\\
&+B\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)+B^2\gamma^j\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\\
&+B\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\\
&+B\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(\frac{\delta^i}{\alpha}-1\right)\\
&-\frac{7}{\delta^i}\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\Bigg(\frac{\delta^i}{\alpha}\\
&-1\Bigg)\Bigg)\frac{1}{\delta^i}S(i)\Bigg)\left(\textit{Αφού }\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\ge\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
\ge\frac{1}{8}\sum_{1\le i\le m_{23}}\sum_{1\le j\le r}\Bigg(\Bigg(&B^3\gamma^j\gamma^j+B^2\gamma^j\left(\frac{\delta^i}{\alpha}-1\right)\\
&+\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\Bigg(2B^2\gamma^j+2B\left(\frac{\delta^i}{\alpha}-1\right)\\
&+B\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\\
&-\frac{7}{\delta^i}\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(\frac{\delta^i}{\alpha}-1\right)\Bigg)\Bigg)\frac{1}{\delta^i}S(i)\Bigg)
\end{aligned}
$$

30. <u>Απλοποίηση της $Δ_{30}$</u>

$$Δ30 = \sum_{1\le k\le r} f^k \sum_{1\le i\le r} f^i \sum_{1\le j\le r} f^j\left(\frac{1}{\gamma^i}-\frac{1}{\gamma^k}\right) - \sum_{1\le k\le r} p_3^k \sum_{1\le i\le r} p_2^i \sum_{1\le j\le r} p_1^j\left(\frac{1}{\gamma^i}-\frac{1}{\gamma^k}\right)$$

$$= \sum_{1\le k<j<i\le r}\left(f^k f^i f^j - p_3^k p_2^i p_1^j\right)\left(\frac{1}{\gamma^i}-\frac{1}{\gamma^k}\right)$$

$$= \frac{1}{8}\sum_{1\le j<k<i\le r}\left(\left(\gamma^k B+\frac{\gamma^k}{\beta}-1\right)\left(\gamma^i A+\frac{\gamma^i}{a}-1\right)\left(\gamma^j B+\frac{\gamma^j}{\beta}-1\right) - \left(\frac{\gamma^k}{\beta}-\frac{\gamma^k}{\alpha}+\frac{\gamma^k}{\gamma}-1\right)\left(\frac{\gamma^i}{a}-\frac{\gamma^i}{\beta}+\frac{\gamma^i}{\gamma}-1\right)\left(\frac{\gamma^j}{\beta}+\frac{\gamma^j}{\alpha}-\frac{\gamma^j}{\gamma}-1\right)\right)\left(\frac{1}{\gamma^i}-\frac{1}{\gamma^k}\right)$$

$$= \frac{1}{8}\sum_{1\le j<k<i\le r}\left(\left(\gamma^k B+\frac{\gamma^k}{\beta}-1\right)\left(\gamma^i A+\frac{\gamma^i}{a}-1\right)\left(\gamma^j B+\frac{\gamma^j}{\beta}-1\right) - \left(\frac{\gamma^k}{\beta}-\frac{\gamma^k}{\alpha}+\frac{\gamma^k}{\gamma}-1\right)\left(\frac{\gamma^i}{a}-\frac{\gamma^i}{\beta}+\frac{\gamma^i}{\gamma}-1\right)\left(\frac{\gamma^j}{\beta}+\frac{\gamma^j}{\alpha}-\frac{\gamma^j}{\gamma}-1\right)\right)\left(\frac{1}{\gamma^i}-\frac{1}{\gamma^k}\right)$$

$$\ge \frac{1}{8}\sum_{1\le j<k<i\le r}\left(\left(\gamma^k B+\frac{\gamma^k}{\beta}-1\right)\left(\gamma^i A+\frac{\gamma^i}{a}-1\right)\left(\gamma^j B+\frac{\gamma^j}{\beta}-1\right) - \left(\frac{\gamma^k}{\beta}-1\right)\left(\frac{\gamma^i}{a}-1\right)\left(\frac{\gamma^j}{\beta}-1\right)\right)\left(\frac{1}{\gamma^i}-\frac{1}{\gamma^k}\right)(Αφού\ \gamma\ge\beta\ge\alpha)$$

31. <u>Απλοποίηση της $Δ_{31}$</u>

$$Δ_{31} = \sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_1} f^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\varepsilon^k}\right) - \sum_{1\le k\le m_{13}} p_3^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{12}} p_2^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_1} p_1^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\varepsilon^k}\right)$$

$$= \sum_{1\le k\le m_{13}} \sum_{1\le i\le m_{12}} \sum_{1\le j\le m_1} \Big(f^{r+m_1+m_2+m_3+m_{12}+k} f^{r+m_1+m_2+m_3+i} f^{r+j} - p_1^{r+j} p_2^{r+m_1+m_2+m_3+i} p_3^{r+m_1+m_2+m_3+m_{12}+k}\Big)\left(\frac{1}{\gamma}-\frac{1}{\varepsilon^k}\right)$$

$$= \sum_{1\le i\le m_{12}} \sum_{1\le k\le m_{13}} \Big(\frac{1}{2}\Big(A\varepsilon^k + p_3^{r+m_1+m_2+m_3+m_{12}+k}\Big)\frac{1}{2}\Big(B\delta^i + p_2^{r+m_1+m_2+m_3+i}\Big) m_1\gamma G - \Big(1-\frac{\Gamma-r\beta}{2\beta}-\frac{\Gamma}{2\alpha}+\frac{\Gamma}{2\gamma}-\frac{Z-m_{12}\beta}{\beta}-\frac{E-m_{13}\alpha}{\alpha}\Big) p_2^{r+m_1+m_2+m_3+i} p_3^{r+m_1+m_2+m_3+m_{12}+k}\Big)\left(\frac{1}{\gamma}-\frac{1}{\varepsilon^k}\right)$$

$$\ge \sum_{1\le i\le m_{12}} \sum_{1\le k\le m_{13}} \Big(\frac{m_1\gamma G}{4} p_3^{r+m_1+m_2+m_3+m_{12}+k} p_2^{r+m_1+m_2+m_3+i} - \Big(1-\frac{\Gamma-r\beta}{2\beta}-\frac{\Gamma}{2\alpha}+\frac{\Gamma}{2\gamma}-\frac{Z-m_{12}\beta}{\beta}-\frac{E-m_{13}\alpha}{\alpha}\Big) p_2^{r+m_1+m_2+m_3+i} p_3^{r+m_1+m_2+m_3+m_{12}+k}\Big)\left(\frac{1}{\gamma}-\frac{1}{\varepsilon^k}\right)$$

$$\ge \left(\frac{m_1\gamma G}{4} - \left(\frac{(r+2)\beta-\Gamma}{2\beta}-\frac{\Gamma}{2\alpha}+\frac{\Gamma}{2\gamma}-\frac{Z-m_{12}\beta}{\beta}-\frac{E-m_{13}\alpha}{\alpha}\right)\right) \sum_{1\le i\le m_{12}} \sum_{1\le k\le m_{13}} \left(p_3^{r+m_1+m_2+m_3+m_{12}+k} p_2^{r+m_1+m_2+m_3+i}\right)\left(\frac{1}{\gamma}-\frac{1}{\varepsilon^k}\right)$$

$$\ge \left(\frac{m_1\gamma G}{4} - \left(1-\frac{\Gamma-r\beta}{2\beta}-\frac{\Gamma}{2\alpha}+\frac{\Gamma}{2\gamma}-\frac{Z-m_{12}\beta}{\beta}-\frac{E-m_{13}\alpha}{\alpha}\right)\right)\frac{1}{\gamma^3} \sum_{1\le k\le m_{13}} \sum_{1\le i\le m_{12}} \left(\left(\zeta^i-\gamma\right)\frac{\left(\varepsilon^k-\gamma\right)^2}{\varepsilon^k}\right)$$

32. <u>Απλοποίηση της Δ_{32}</u>

$$\Delta_{32} = \sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le r} f^i \sum_{1\le j\le m_1} f^{jr+}\left(\frac{1}{\gamma}-\frac{1}{\varepsilon^k}\right) - \sum_{1\le k\le m_{13}} p_3^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le r} p_2^i \sum_{1\le j\le m_1} p_1^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\varepsilon^k}\right)$$

$$= \sum_{1\le k\le m_{13}} \sum_{1\le i\le r} \sum_{1\le j\le m_1} \left(f^{r+m_1+m_2+m_3+m_{12}+k} f^i \gamma G - p_3^{r+m_1+m_2+m_3+m_{12}+k} p_2^i \left(1-\frac{\Gamma-r\beta}{2\beta}-\frac{\Gamma}{2\alpha}+\frac{\Gamma}{2\gamma}-\frac{Z-m_{12}\beta}{\beta}-\frac{E-m_{13}\alpha}{\alpha}\right)\right)\left(\frac{1}{\gamma}-\frac{1}{\varepsilon^k}\right)$$

$$= \sum_{1\le k\le m_{13}} \sum_{1\le i\le r} \left(\frac{1}{2}\left(A\varepsilon^k + p_3^{r+m_1+m_2+m_3+m_{12}+k}\right)\frac{1}{2}\left(\gamma^i A + \frac{\gamma^i}{a}-1\right) m_1\gamma G - \left(\frac{\varepsilon^k}{\gamma}-1\right)\left(\frac{\gamma^i}{2a}-\frac{\gamma^i}{2\beta}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\left(1-\frac{\Gamma-r\beta}{2\beta}-\frac{\Gamma}{2\alpha}+\frac{\Gamma}{2\gamma}-\frac{Z-m_{12}\beta}{\beta}-\frac{E-m_{13}\alpha}{\alpha}\right)\right)\left(\frac{1}{\gamma}-\frac{1}{\varepsilon^k}\right)$$

$$\geq \frac{1}{2}\sum_{1\leq k\leq m_{13}}\sum_{1\leq i\leq r}\left(p_3^{r+m_1+m_2+m_3+m_{12}+k}\left(\frac{\gamma^i}{a}-1\right)\frac{m_1\gamma G}{2}\right.$$

$$-p_3^{r+m_1+m_2+m_3+m_{12}+k}\left(\frac{\gamma^i}{a}-1\right)\left(1-\frac{\Gamma-r\beta}{2\beta}-\frac{\Gamma}{2\alpha}+\frac{\Gamma}{2\gamma}-\frac{Z-m_{12}\beta}{\beta}\right.$$

$$\left.\left.-\frac{E-m_{13}\alpha}{\alpha}\right)\right)\left(\frac{1}{\gamma}-\frac{1}{\varepsilon^k}\right)$$

$$=\left(\frac{m_1\gamma G}{4}-\left(1-\frac{\Gamma-r\beta}{2\beta}-\frac{\Gamma}{2\alpha}+\frac{\Gamma}{2\gamma}-\frac{Z-m_{12}\beta}{\beta}\right.\right.$$

$$\left.\left.-\frac{E-m_{13}\alpha}{\alpha}\right)\right)\frac{1}{\alpha\gamma^2}\sum_{1\leq k\leq m_{13}}\sum_{1\leq i\leq r}\left(\left(\gamma^i-\alpha\right)\left(\frac{(\varepsilon^k-\gamma)^2}{\varepsilon^k}\right)\right)$$

33. <u>Απλοποίηση της Δ_{33}</u>

$$\Delta_{33}=\sum_{1\leq k\leq m_{23}}f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k}\sum_{1\leq i\leq m_{12}}f^{r+m_1+m_2+m_3+i}\sum_{1\leq j\leq m_1}f^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\delta^k}\right)$$

$$-\sum_{1\leq k\leq m_{23}}p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k}\sum_{1\leq i\leq m_{12}}p_2^{r+m_1+m_2+m_3+m_{12}+i}\sum_{1\leq j\leq m_1}p_1^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\delta^k}\right)$$

$$=\sum_{1\leq k\leq m_{23}}\sum_{1\leq i\leq m_{12}}\sum_{1\leq j\leq m_1}\left(f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k}f^{r+m_1+m_2+m_3+i}f^{r+j}\right.$$

$$\left.-p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k}p_2^{r+m_1+m_2+m_3+m_{12}+i}p_1^{r+j}\right)\left(\frac{1}{\gamma}-\frac{1}{\delta^k}\right)$$

$$\geq\left(\frac{m_1\gamma G}{4}-\left(1-\frac{\Gamma-r\beta}{2\beta}-\frac{\Gamma}{2\alpha}+\frac{\Gamma}{2\gamma}-\frac{Z-m_{12}\beta}{\beta}\right.\right.$$

$$\left.\left.-\frac{E-m_{13}\alpha}{\alpha}\right)\right)\sum_{1\leq k\leq m_{23}}\sum_{1\leq i\leq m_{12}}\left(\left(\frac{\delta^k}{\beta}-1\right)\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{1}{\gamma}-\frac{1}{\delta^k}\right)\right)$$

$$=\left(\frac{m_1\gamma G}{4}-\left(1-\frac{\Gamma-r\beta}{2\beta}-\frac{\Gamma}{2\alpha}+\frac{\Gamma}{2\gamma}-\frac{Z-m_{12}\beta}{\beta}\right.\right.$$

$$\left.\left.-\frac{E-m_{13}\alpha}{\alpha}\right)\right)\sum_{1\leq k\leq m_{23}}\sum_{1\leq i\leq m_{12}}\left(\left(\frac{\delta^k}{\beta}-1\right)\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{1}{\gamma}-\frac{1}{\delta^k}\right)\right)$$

$$
=\left(\frac{m_1\gamma G}{4}-\left(1-\frac{\Gamma-r\beta}{2\beta}-\frac{\Gamma}{2\alpha}+\frac{\Gamma}{2\gamma}-\frac{Z-m_{12}\beta}{\beta}\right.\right.
$$

$$
\left.\left.-\frac{E-m_{13}\alpha}{\alpha}\right)\right)\frac{1}{\beta\gamma^2}\sum_{1\le k\le m_{23}}\sum_{1\le i\le m_{12}}\left((\delta^k-\beta)(\zeta^i-\gamma)\left(\frac{\delta^k-\gamma}{\delta^k}\right)\right)
$$

34. Απλοποίηση της Δ_{34}

$$
\Delta_{34}=\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k}\sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i}\sum_{1\le j\le m_1} f^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\delta^k}\right)
$$

$$
-\sum_{1\le k\le m_{23}} p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k}\sum_{1\le i\le m_{23}} p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i}\sum_{1\le j\le m_1} p_1^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\delta^k}\right)
$$

$$
=\sum_{1\le i<k\le m_{23}}\left(\frac{1}{2}\left(\delta^k B+p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k}\right)\frac{1}{2}\left(B\delta^i+p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i}\right)\gamma G\right.
$$

$$
\left.-p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i}\left(1-\frac{\Gamma-r\alpha}{2\alpha}+\frac{\Gamma}{2\beta}-\frac{\Gamma}{2\gamma}-\frac{Z-m_{12}\gamma}{\gamma}-\frac{\Delta-m_{23}\alpha}{\alpha}\right)\right)\left(\frac{1}{\gamma}-\frac{1}{\delta^k}\right)
$$

$$
\ge\left(\frac{m_1\gamma G}{4}-\left(1-\frac{\Gamma-r\alpha}{2\alpha}+\frac{\Gamma}{2\beta}-\frac{\Gamma}{2\gamma}-\frac{Z-m_{12}\gamma}{\gamma}-\frac{\Delta-m_{23}\alpha}{\alpha}\right)\right)\sum_{1\le i<k\le m_{23}}\left(\left(\frac{\delta^k}{\beta}-1\right)\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{1}{\gamma}-\frac{1}{\delta^k}\right)\right)
$$

$$
=\left(\frac{m_1\gamma G}{4}-\left(1-\frac{\Gamma-r\alpha}{2\alpha}+\frac{\Gamma}{2\beta}-\frac{\Gamma}{2\gamma}-\frac{Z-m_{12}\gamma}{\gamma}\right.\right.
$$

$$
\left.\left.-\frac{\Delta-m_{23}\alpha}{\alpha}\right)\right)\frac{1}{\alpha\beta\gamma}\sum_{1\le i<k\le m_{23}}\left((\delta^k-\beta)(\delta^i-\alpha)\left(\frac{\delta^k-\gamma}{\delta^k}\right)\right)
$$

35. Απλοποίηση της Δ_{35}

$$
\Delta_{35}=\sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k}\sum_{1\le i\le r} f^i\sum_{1\le j\le m_1} f^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\delta^k}\right)
$$

$$
-\sum_{1\le k\le m_{23}} p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k}\sum_{1\le i\le r} p_2^i\sum_{1\le j\le m_1} p_1^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\delta^k}\right)
$$

$$
=\sum_{1\le k\le m_{23}}\sum_{1\le i\le r}\sum_{1\le j\le m_1}\left(f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k}f^i f^{r+j}-p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k}p_2^i p_1^{r+j}\right)\left(\frac{1}{\gamma}\right.
$$

$$
\left.-\frac{1}{\delta^k}\right)
$$

$$= \sum_{1\le k\le m_{23}} \sum_{1\le i\le r} \left(\frac{1}{2}\left(\delta^k B + p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k}\right)\frac{1}{2}\left(\gamma^i A + \frac{\gamma^i}{a} - 1\right) m_1\gamma G - p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} p_2^i p_1^{r+j}\right)\left(\frac{1}{\gamma} - \frac{1}{\delta^k}\right)$$

$$= \sum_{1\le k\le m_{23}} \sum_{1\le i\le r} \left(\frac{m_1\gamma G}{4}\left(\delta^k B + p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k}\right)\left(\gamma^i A + \frac{\gamma^i}{a} - 1\right) - p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} p_2^i \left(1 - \frac{\Gamma - r\alpha}{2\alpha} + \frac{\Gamma}{2\beta} - \frac{\Gamma}{2\gamma} - \frac{Z - m_{12}\gamma}{\gamma} - \frac{\Delta - m_{23}\alpha}{\alpha}\right)\right)\left(\frac{1}{\gamma} - \frac{1}{\delta^k}\right)$$

$$\ge \sum_{1\le k\le m_{23}} \sum_{1\le i\le r} \left(\frac{m_1\gamma G}{4}\left(p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k}\right)\left(\frac{\gamma^i}{a} - 1\right) - p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k}\left(\frac{\gamma^i}{a} - 1\right)\left(1 - \frac{\Gamma - r\alpha}{2\alpha} + \frac{\Gamma}{2\beta} - \frac{\Gamma}{2\gamma} - \frac{Z - m_{12}\gamma}{\gamma} - \frac{\Delta - m_{23}\alpha}{\alpha}\right)\right)\left(\frac{1}{\gamma} - \frac{1}{\delta^k}\right)$$

$$= \left(\frac{m_1\gamma G}{4} - \left(1 - \frac{\Gamma - r\alpha}{2\alpha} + \frac{\Gamma}{2\beta} - \frac{\Gamma}{2\gamma} - \frac{Z - m_{12}\gamma}{\gamma} - \frac{\Delta - m_{23}\alpha}{\alpha}\right)\right)\frac{1}{\beta} \sum_{1\le k\le m_{23}} \sum_{1\le i\le r} \left(\left(\frac{\delta^k}{\beta} - 1\right)\left(\frac{\gamma^i}{a} - 1\right)\right)\left(\frac{1}{\gamma} - \frac{1}{\delta^k}\right)$$

$$= \left(\frac{m_1\gamma G}{4} - \left(1 - \frac{\Gamma - r\alpha}{2\alpha} + \frac{\Gamma}{2\beta} - \frac{\Gamma}{2\gamma} - \frac{Z - m_{12}\gamma}{\gamma} - \frac{\Delta - m_{23}\alpha}{\alpha}\right)\right)\frac{1}{\alpha\beta\gamma} \sum_{1\le k\le m_{23}} \sum_{1\le i\le r} \left((\delta^k - \beta)(\gamma^i - \alpha)\frac{(\delta^k - \gamma)}{\delta^k}\right)$$

$= \left(\frac{m_1\gamma G}{4} - \left(1 - \frac{\Gamma - r\alpha}{2\alpha} + \frac{\Gamma}{2\beta} - \frac{\Gamma}{2\gamma} - \frac{Z - m_{12}\gamma}{\gamma} - \frac{\Delta - m_{23}\alpha}{\alpha}\right)\right)\frac{1}{\alpha\beta\gamma} \sum_{1\le k\le m_{23}} \sum_{1\le i\le r}\left((\delta^k - \beta)(\gamma^i - \alpha)\frac{(\delta^k - \gamma)}{\delta^k}\right)$

36. <u>Απλοποίηση της Δ_{36}</u>

$$\varDelta_{36} = \sum_{1\le k\le r} f^k \sum_{1\le i\le r} f^i \sum_{1\le j\le m_1} f^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\gamma^k}\right) - \sum_{1\le k\le r} p_3^k \sum_{1\le i\le r} p_2^i \sum_{1\le j\le m_1} p_1^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\gamma^k}\right)$$

$$\ge \left(\frac{m_1\gamma G}{4} - \left(\frac{(r+2)\beta-\varGamma}{2\beta} - \frac{Z-m_{12}\beta}{\beta} - \frac{E-m_{13}\alpha}{\alpha} + \frac{\varGamma(\alpha-\gamma)}{2\beta}\right)\right)\sum_{1\le k<i\le r}\left(\left(\frac{\gamma^k}{\beta}-1\right)\left(\frac{\gamma^i}{a}-1\right)\right)\left(\frac{1}{\gamma}-\frac{1}{\gamma^k}\right)$$

$$= \left(\frac{m_1\gamma G}{4} - \left(\frac{(r+2)\beta-\varGamma}{2\beta} - \frac{Z-m_{12}\beta}{\beta} - \frac{E-m_{13}\alpha}{\alpha} + \frac{\varGamma(\alpha-\gamma)}{2\beta}\right)\right)\frac{1}{\alpha\beta\gamma}\sum_{1\le k<i\le r}\left((\gamma^k-\beta)(\gamma^i-\alpha)\left(\frac{\gamma^k-\gamma}{\gamma^k}\right)\right)$$

37. <u>Απλοποίηση της Δ_{37}</u>

$$\varDelta_{37} = \sum_{1\le k\le r} f^k \sum_{1\le i\le r} f^i \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j}\left(\frac{1}{\zeta^j}-\frac{1}{\gamma^k}\right) - \sum_{1\le k\le r} p_3^k \sum_{1\le i\le r} p_2^i \sum_{1\le j\le m_{12}} p_1^{r+m_1+m_2+m_3+j}\left(\frac{1}{\zeta^j}-\frac{1}{\gamma^k}\right)$$

$$= \sum_{1\le k\le r}\sum_{1\le i\le r}\sum_{1\le j\le m_{12}}\left(f^k f^i f^{r+m_1+m_2+m_3+j} - p_3^k p_2^i p_1^{r+m_1+m_2+m_3+j}\right)\left(\frac{1}{\zeta^j}-\frac{1}{\gamma^k}\right)$$

$$= \sum_{1\le k\le r}\sum_{1\le i\le r}\sum_{1\le j\le m_{12}}\left(\frac{1}{2}\left(\gamma^k B + \frac{\gamma^k}{\beta}-1\right)\frac{1}{2}\left(\gamma^i A + \frac{\gamma^i}{a}-1\right)\frac{1}{2}\left(B\zeta^j + \frac{\zeta^j}{\beta}-1\right) - \left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\gamma^i}{2a}-\frac{\gamma^i}{2\beta}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\left(\frac{\zeta^j}{\beta}-1\right)\right)\left(\frac{1}{\zeta^j}-\frac{1}{\gamma^k}\right)$$

$$\geq \sum_{1\leq i<k\leq r} \sum_{1\leq j\leq m_{12}} \left(\frac{1}{2}\left(\gamma^k B + p_3^k\right)\frac{1}{2}\left(\gamma^i A + p_2^i\right)\frac{1}{2}\left(B\zeta^j + p_1^j\right) - p_3^k p_2^i p_1^j\right)\left(\frac{1}{\zeta^j} - \frac{1}{\gamma^k}\right)\left(A\varphi o\acute{\upsilon}\ \frac{\gamma^i}{a} - 1 > p_2^i\ \kappa\alpha\iota\ \frac{\gamma^k}{\beta} - 1 > p_3^k\right)$$

$$= \sum_{1\leq i<k\leq r} \sum_{1\leq j\leq m_{12}} \left(\frac{1}{8}\left(\gamma^k B\gamma^i A + \gamma^k B p_2^i + p_3^k\gamma^i A + p_3^k p_2^i\right)\left(B\zeta^j + p_1^j\right) - p_3^k p_2^i p_1^j\right)\left(\frac{1}{\zeta^j} - \frac{1}{\gamma^k}\right)$$

$$= \sum_{1\leq i<k\leq r} \sum_{1\leq j\leq m_{12}} \left(\frac{1}{8}\left(\gamma^k B\gamma^i AB\zeta^j + \gamma^k B\gamma^i A p_1^j + \gamma^k B p_2^i B\zeta^j + \gamma^k B p_2^i p_1^j + p_3^k\gamma^i AB\zeta^j + p_3^k\gamma^i A p_1^j + p_3^k p_2^i B\zeta^j - 7p_3^k p_2^i p_1^j\right)\right)\left(\frac{1}{\zeta^j} - \frac{1}{\gamma^k}\right)$$

$$\geq \sum_{1\leq i<k\leq r} \sum_{1\leq j\leq m_{12}} \Bigg(\frac{1}{8}\Bigg(A^3\gamma^k\gamma^i\zeta^j + A^2\gamma^k\gamma^i\left(\frac{\zeta^j}{\beta} - 1\right) + A^2\gamma^k\zeta^j\left(\frac{\gamma^i}{2a} - \frac{\gamma^i}{2\beta} + \frac{\gamma^i}{2\gamma} - \frac{1}{2}\right)$$
$$+ A\gamma^k\left(\frac{\gamma^i}{2a} - \frac{\gamma^i}{2\beta} + \frac{\gamma^i}{2\gamma} - \frac{1}{2}\right)\left(\frac{\zeta^j}{\beta} - 1\right) + A^2\gamma^i\zeta^j\left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2}\right)$$
$$+ A\gamma^i\left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2}\right)\left(\frac{\zeta^j}{\beta} - 1\right)$$
$$+ A\zeta^j\left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2}\right)\left(\frac{\gamma^i}{2a} - \frac{\gamma^i}{2\beta} + \frac{\gamma^i}{2\gamma} - \frac{1}{2}\right)$$
$$- 7\left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2}\right)\left(\frac{\gamma^i}{2a} - \frac{\gamma^i}{2\beta} + \frac{\gamma^i}{2\gamma} - \frac{1}{2}\right)\left(\frac{\zeta^j}{\beta} - 1\right)\Bigg)\Bigg)\left(\frac{1}{\zeta^j} - \frac{1}{\gamma^k}\right)\ (A\varphi o\acute{\upsilon}\ B \geq A)$$

$$= \sum_{1\leq i\leq r} \sum_{1\leq j\leq m_{12}} \Bigg(\frac{1}{8}\Bigg(A^3\gamma^i\zeta^j + A^2\gamma^i\left(\frac{\zeta^j}{\beta} - 1\right) + A^2\zeta^j\left(\frac{\gamma^i}{2a} - \frac{\gamma^i}{2\beta} + \frac{\gamma^i}{2\gamma} - \frac{1}{2}\right)$$
$$+ A\left(\frac{\gamma^i}{2a} - \frac{\gamma^i}{2\beta} + \frac{\gamma^i}{2\gamma} - \frac{1}{2}\right)\left(\frac{\zeta^j}{\beta} - 1\right) + A^2\zeta^j\left(\frac{\gamma^i}{2\beta} - \frac{\gamma^i}{2\alpha} + \frac{\gamma^i}{2\gamma} - \frac{1}{2}\right)$$
$$+ A\left(\frac{\gamma^i}{2\beta} - \frac{\gamma^i}{2\alpha} + \frac{\gamma^i}{2\gamma} - \frac{1}{2}\right)\left(\frac{\zeta^j}{\beta} - 1\right)$$
$$+ A\left(\frac{\gamma^i}{2\beta} - \frac{\gamma^i}{2\alpha} + \frac{\gamma^i}{2\gamma} - \frac{1}{2}\right)\left(\frac{\gamma^i}{2a} - \frac{\gamma^i}{2\beta} + \frac{\gamma^i}{2\gamma} - \frac{1}{2}\right)$$
$$- \frac{7}{\zeta^j}\left(\frac{\gamma^i}{2\beta} - \frac{\gamma^i}{2\alpha} + \frac{\gamma^i}{2\gamma} - \frac{1}{2}\right)\left(\frac{\gamma^i}{2a} - \frac{\gamma^i}{2\beta} + \frac{\gamma^i}{2\gamma} - \frac{1}{2}\right)\left(\frac{\zeta^j}{\beta} - 1\right)\Bigg)\sum_{1\leq k\leq r}\left(\frac{\gamma^k}{\zeta^j} - 1\right)\Bigg)$$

$$\geq \sum_{1\leq i\leq r}\sum_{1\leq j\leq m_{12}}\left(\frac{1}{8}\left(A^3\gamma^i\varepsilon^j + A^2\gamma^i\left(\frac{\zeta^j}{\beta}-1\right) + A^2\varepsilon^j\left(\frac{\gamma^i}{2\beta}-\frac{\gamma^i}{2\alpha}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\right.\right.$$

$$+ A\left(\frac{\gamma^i}{2\beta}-\frac{\gamma^i}{2\alpha}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\left(\frac{\zeta^j}{\beta}-1\right) + A^2\zeta^j\left(\frac{\gamma^i}{2\beta}-\frac{\gamma^i}{2\alpha}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)$$

$$+ A\left(\frac{\gamma^i}{2\beta}-\frac{\gamma^i}{2\alpha}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\left(\frac{\zeta^j}{\beta}-1\right)$$

$$+ A\left(\frac{\gamma^i}{2\beta}-\frac{\gamma^i}{2\alpha}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\left(\frac{\gamma^i}{2\beta}-\frac{\gamma^i}{2\alpha}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)$$

$$\left.-\frac{7}{\zeta^j}\left(\frac{\gamma^i}{2\beta}-\frac{\gamma^i}{2\alpha}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\left(\frac{\gamma^i}{2\beta}-\frac{\gamma^i}{2\alpha}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\left(\frac{\zeta^j}{\beta}-1\right)\right)\frac{1}{\zeta^j}\sum_{1\leq k\leq r}(\gamma^k$$

$$\left.-\zeta^j)\right)(A\varphi o\acute{\upsilon}\ \beta \geq \alpha)$$

$$= \sum_{1\leq i\leq r}\sum_{1\leq j\leq m_{12}}\left(\frac{1}{8}\left(A^3\gamma^i\zeta^j + A^2\gamma^i\left(\frac{\zeta^j}{\beta}-1\right)\right.\right.$$

$$+\left(\frac{\gamma^i}{2\beta}-\frac{\gamma^i}{2\alpha}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\left(2A^2\zeta^j + 2A\left(\frac{\zeta^j}{\beta}-1\right) + A\left(\frac{\gamma^i}{2\beta}-\frac{\gamma^i}{2\alpha}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\right.$$

$$\left.\left.\left.-\frac{7}{\zeta^j}\left(\frac{\gamma^i}{2\beta}-\frac{\gamma^i}{2\alpha}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\left(\frac{\zeta^j}{\beta}-1\right)\right)\right)\frac{1}{\zeta^j}S(j)\right)$$

38. <u>Απλοποίηση της Δ_{38}</u>

$$\Delta_{38} = \sum_{1\leq k\leq m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k}\sum_{1\leq i\leq m_{12}} f^{r+m_1+m_2+m_3+i}\sum_{1\leq j\leq m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\varepsilon^j}-\frac{1}{\varepsilon^k}\right)$$

$$-\sum_{1\leq k\leq m_{13}} p_3^{r+m_1+m_2+m_3+m_{12}+k}\sum_{1\leq i\leq m_{12}} p_2^{r+m_1+m_2+m_3+i}\sum_{1\leq j\leq m_{13}} p_1^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\varepsilon^j}-\frac{1}{\varepsilon^k}\right)$$

$$= \sum_{1\leq k\leq m_{13}}\sum_{1\leq i\leq m_{12}}\sum_{1\leq j\leq m_{13}}\left(f^{r+m_1+m_2+m_3+m_{12}+k}f^{r+m_1+m_2+m_3+i}f^{r+m_1+m_2+m_3+m_{12}+j}\right.$$

$$\left.-p_3^{r+m_1+m_2+m_3+m_{12}+k}p_2^{r+m_1+m_2+m_3+i}p_1^{r+m_1+m_2+m_3+m_{12}+j}\right)\left(\frac{1}{\varepsilon^j}-\frac{1}{\varepsilon^k}\right)$$

$$= \sum_{1\le i\le m_{12}} \sum_{1\le k<j\le m_{13}} \left(\frac{1}{2}\left(G\varepsilon^k + p_3^{r+m_1+m_2+m_3+m_{12}+k}\right)\frac{1}{2}\left(G\zeta^i + p_2^{r+m_1+m_2+m_3+i}\right)\frac{1}{2}\left(A\varepsilon^j + p_1^{r+m_1+m_2+m_3+m_{12}+j}\right) - p_3^{r+m_1+m_2+m_3+m_{12}+k} p_2^{r+m_1+m_2+m_3+i} p_1^{r+m_1+m_2+m_3+m_{12}+j}\right)\left(\frac{1}{\varepsilon^j} - \frac{1}{\varepsilon^k}\right)$$

$$= \sum_{1\le i\le m_{12}} \sum_{1\le k<j\le m_{13}} \left(\frac{1}{4}\left(G\varepsilon^k G\zeta^i + G\varepsilon^k p_2^i + G\zeta^i p_3^k + p_3^k p_2^i\right)\frac{1}{2}\left(A\varepsilon^j + p_1^j\right) - p_3^{r+m_1+m_2+m_3+m_{12}+k} p_2^{r+m_1+m_2+m_3+i} p_1^{r+m_1+m_2+m_3+m_{12}+j}\right)\left(\frac{1}{\varepsilon^j} - \frac{1}{\varepsilon^k}\right)$$

$$= \sum_{1\le i\le m_{12}} \sum_{1\le k<j\le m_{13}} \left(\frac{1}{8}\left(G\varepsilon^k G\zeta^i A\varepsilon^j + G\varepsilon^k G\zeta^i p_1^j + G\varepsilon^k p_2^i A\varepsilon^j + G\varepsilon^k p_2^i p_1^j + G\zeta^i p_3^k A\varepsilon^j + G\zeta^i p_3^k p_1^j + p_3^k p_2^i A\varepsilon^j - 7 p_3^k p_2^i p_1^j\right)\right)\left(\frac{1}{\varepsilon^j} - \frac{1}{\varepsilon^k}\right)$$

$$\ge \sum_{1\le i\le m_{12}} \sum_{1\le k<j\le m_{13}} \left(\frac{1}{8}\left(A^3\varepsilon^k\zeta^i\varepsilon^j + A^2\varepsilon^k\zeta^i\left(\frac{\varepsilon^j}{\alpha}-1\right) + A^2\varepsilon^k\varepsilon^j\left(\frac{\zeta^i}{\gamma}-1\right) + A\varepsilon^k\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{\varepsilon^j}{\alpha}-1\right) + A^2\zeta^i\varepsilon^j\left(\frac{\varepsilon^k}{\gamma}-1\right) + A\zeta^i\left(\frac{\varepsilon^k}{\gamma}-1\right)\left(\frac{\varepsilon^j}{\alpha}-1\right) + A\varepsilon^j\left(\frac{\varepsilon^k}{\gamma}-1\right)\left(\frac{\zeta^i}{\gamma}-1\right) - 7\left(\frac{\varepsilon^k}{\gamma}-1\right)\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{\varepsilon^j}{\alpha}-1\right)\right)\right)\left(\frac{1}{\varepsilon^j} - \frac{1}{\varepsilon^k}\right) (\text{Αφού } G \ge A)$$

$$\ge \sum_{1\le i\le m_{12}} \sum_{1\le k\le m_{13}} \left(\frac{1}{8}\left(A^3\varepsilon^k\zeta^i + A^2\zeta^i\left(\frac{\varepsilon^k}{\gamma}-1\right) + A^2\varepsilon^k\left(\frac{\zeta^i}{\gamma}-1\right) + A\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{\varepsilon^k}{\gamma}-1\right) + A^2\zeta^i\left(\frac{\varepsilon^k}{\gamma}-1\right) + A\zeta^i\left(\frac{\varepsilon^k}{\gamma}-1\right)\left(\frac{\varepsilon^k}{\gamma}-1\right) + A\left(\frac{\varepsilon^k}{\gamma}-1\right)\left(\frac{\zeta^i}{\gamma}-1\right) - \frac{7}{\varepsilon^k}\left(\frac{\varepsilon^k}{\gamma}-1\right)\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{\varepsilon^k}{\alpha}-1\right)\right)\sum_{1\le j\le m_{13}}\left(1-\frac{\varepsilon^j}{\varepsilon^k}\right)\right) (\text{Αφού } \gamma \ge \alpha)$$

$$=\frac{1}{8}\sum_{1\le i\le m_{12}}\sum_{1\le k\le m_{13}}\Bigg(A^3\varepsilon^k\zeta^i+A^2\varepsilon^k\left(\frac{\zeta^i}{\gamma}-1\right)$$
$$+\left(\frac{\varepsilon^k}{\gamma}-1\right)\Bigg(2A^2\zeta^i+2A\left(\frac{\zeta^i}{\gamma}-1\right)+A\zeta^i\left(\frac{\varepsilon^k}{\gamma}-1\right)$$
$$-\frac{7}{\varepsilon^k}\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{\varepsilon^k}{\alpha}-1\right)\Bigg)\frac{1}{\varepsilon^k}S(k)\Bigg)$$

39. <u>Απλοποίηση της Δ_{39}</u>

$$\Delta_{39}=\sum_{1\le k\le m_{13}}f^{r+m_1+m_2+m_3+m_{12}+k}\sum_{1\le i\le r}f^i\sum_{1\le j\le m_{13}}f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\varepsilon^j}-\frac{1}{\varepsilon^k}\right)$$
$$-\sum_{1\le k\le m_{13}}p_3^{r+m_1+m_2+m_3+m_{12}+k}\sum_{1\le i\le r}p_2^i\sum_{1\le j\le m_{13}}p_1^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\varepsilon^j}-\frac{1}{\varepsilon^k}\right)$$

$$=\sum_{1\le k\le m_{13}}\sum_{1\le i\le r}\sum_{1\le j\le m_{13}}\Bigg(\frac{1}{2}\left(A\varepsilon^k+p_3^{r+m_1+m_2+m_3+m_{12}+k}\right)\frac{1}{2}\left(\gamma^iA+\frac{\gamma^i}{a}-1\right)\frac{1}{2}\Big(A\varepsilon^j$$
$$+p_1^{r+m_1+m_2+m_3+m_{12}+j}\Big)-p_3^{r+m_1+m_2+m_3+m_{12}+k}p_2^ip_1^{r+m_1+m_2+m_3+m_{12}+j}\Bigg)\left(\frac{1}{\varepsilon^j}-\frac{1}{\varepsilon^k}\right)$$

$$\ge\sum_{1\le k\le m_{13}}\sum_{1\le i\le r}\sum_{1\le j\le m_{13}}\Bigg(\frac{1}{2}\left(A\varepsilon^k+p_3^{r+m_1+m_2+m_3+m_{12}+k}\right)\frac{1}{2}\left(\gamma^iA+p_2^i\right)\frac{1}{2}\left(A\varepsilon^j+p_1^{r+m_1+m_2+m_3+m_{12}+j}\right)$$
$$-p_3^{r+m_1+m_2+m_3+m_{12}+k}p_2^ip_1^{r+m_1+m_2+m_3+m_{12}+j}\Bigg)\left(\frac{1}{\varepsilon^j}-\frac{1}{\varepsilon^k}\right)\left(\text{Αφού }\frac{\gamma^i}{a}-1\ge p_2^i\right)$$

$$=\sum_{1\le k\le m_{13}}\sum_{1\le i\le r}\sum_{1\le j\le m_{13}}\left(\frac{1}{4}\left(A\varepsilon^k\gamma^iA+A\varepsilon^kp_2^i+\gamma^iAp_3^k+p_3^kp_2^i\right)\frac{1}{2}\left(A\varepsilon^j+p_1^j\right)-p_3^kp_2^ip_1^j\right)\left(\frac{1}{\varepsilon^j}-\frac{1}{\varepsilon^k}\right)$$

$$=\sum_{1\le k\le m_{13}}\sum_{1\le i\le r}\sum_{1\le j\le m_{13}}\Bigg(\frac{1}{8}\Big(A\varepsilon^k\gamma^iAA\varepsilon^j+A\varepsilon^k\gamma^iAp_1^j+A\varepsilon^kp_2^iA\varepsilon^j+A\varepsilon^kp_2^ip_1^j+\gamma^iAp_3^kA\varepsilon^j+\gamma^iAp_3^kp_1^j$$
$$+p_3^kp_2^iA\varepsilon^j-7p_3^kp_2^ip_1^j\Big)\Bigg)\left(\frac{1}{\varepsilon^j}-\frac{1}{\varepsilon^k}\right)$$

$$= \sum_{1\le i\le r}\sum_{1\le k<j\le m_{13}}\Bigg(\frac{1}{8}\Bigg(A^3\varepsilon^k\gamma^i\varepsilon^j + A^2\varepsilon^k\gamma^i\left(\frac{\varepsilon^j}{\alpha}-1\right) + A^2\varepsilon^k\varepsilon^j\left(\frac{\gamma^i}{2a}-\frac{\gamma^i}{2\beta}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)$$

$$+ A\varepsilon^k\left(\frac{\gamma^i}{2a}-\frac{\gamma^i}{2\beta}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\left(\frac{\varepsilon^j}{\alpha}-1\right) + A^2\gamma^i\varepsilon^j\left(\frac{\varepsilon^k}{\gamma}-1\right) + A\gamma^i\left(\frac{\varepsilon^k}{\gamma}-1\right)\left(\frac{\varepsilon^j}{\alpha}-1\right)$$

$$+ A\varepsilon^j\left(\frac{\varepsilon^k}{\gamma}-1\right)\left(\frac{\gamma^i}{2a}-\frac{\gamma^i}{2\beta}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)$$

$$- 7\left(\frac{\varepsilon^k}{\gamma}-1\right)\left(\frac{\gamma^i}{2a}-\frac{\gamma^i}{2\beta}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\left(\frac{\varepsilon^j}{\alpha}-1\right)\Bigg)\Bigg)\left(\frac{1}{\varepsilon^j}-\frac{1}{\varepsilon^k}\right)$$

$$\ge \frac{1}{8}\sum_{1\le i\le r}\sum_{1\le k\le m_{13}}\Bigg(\Bigg(A^3\varepsilon^k\gamma^i + A^2\gamma^i\left(\frac{\varepsilon^k}{\gamma}-1\right) + A^2\varepsilon^k\left(\frac{\gamma^i}{2a}-\frac{\gamma^i}{2\beta}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)$$

$$+ A\left(\frac{\gamma^i}{2a}-\frac{\gamma^i}{2\beta}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\left(\frac{\varepsilon^k}{\gamma}-1\right) + A^2\gamma^i\left(\frac{\varepsilon^k}{\gamma}-1\right) + A\gamma^i\left(\frac{\varepsilon^k}{\gamma}-1\right)\left(\frac{\varepsilon^k}{\gamma}-1\right)$$

$$+ A\left(\frac{\varepsilon^k}{\gamma}-1\right)\left(\frac{\gamma^i}{2a}-\frac{\gamma^i}{2\beta}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)$$

$$- \frac{7}{\varepsilon^k}\left(\frac{\varepsilon^k}{\gamma}-1\right)\left(\frac{\gamma^i}{2a}-\frac{\gamma^i}{2\beta}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\left(\frac{\varepsilon^j}{\alpha}-1\right)\Bigg)\sum_{1\le j\le m_{13}}\left(1-\frac{\varepsilon^j}{\varepsilon^k}\right)\Bigg) \quad (A\varphi o\acute{\upsilon}\ \gamma \ge \alpha)$$

$$\ge \frac{1}{8}\sum_{1\le i\le r}\sum_{1\le k\le m_{13}}\Bigg(A^3\varepsilon^k\gamma^i + A^2\varepsilon^k\left(\frac{\gamma^i}{2a}-\frac{\gamma^i}{2\beta}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)$$

$$+ \left(\frac{\varepsilon^k}{\gamma}-1\right)\Bigg(2A^2\gamma^i + 2A\left(\frac{\gamma^i}{2a}-\frac{\gamma^i}{2\beta}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right) + A\gamma^i\left(\frac{\varepsilon^k}{\gamma}-1\right)$$

$$- \frac{7}{\varepsilon^k}\left(\frac{\gamma^i}{2a}-\frac{\gamma^i}{2\beta}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\left(\frac{\varepsilon^k}{\alpha}-1\right)\Bigg)\frac{1}{\varepsilon^k}S(k)\Bigg)$$

40. <u>Απλοποίηση της Δ_{40}</u>

$$\Delta_{40} = \sum_{1\le k\le r} f^k \sum_{1\le i\le r} f^i \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\varepsilon^j}-\frac{1}{\gamma^k}\right)$$

$$- \sum_{1\le k\le r} p_3^k \sum_{1\le i\le r} p_2^i \sum_{1\le j\le m_{13}} p_1^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\varepsilon^j}-\frac{1}{\gamma^k}\right)$$

$$= \sum_{1\le i<k\le r} \sum_{1\le j\le m_{13}} \left(f^k f^i f^{r+m_1+m_2+m_3+m_{12}+j} - p_3^k p_2^i p_1^{r+m_1+m_2+m_3+m_{12}+j}\right)\left(\frac{1}{\varepsilon^j} - \frac{1}{\gamma^k}\right)$$

$$= \sum_{1\le i<k\le r} \sum_{1\le j\le m_{13}} \left(\frac{1}{2}\left(\gamma^k B + \frac{\gamma^k}{\beta} - 1\right)\frac{1}{2}\left(\gamma^i A + \frac{\gamma^i}{a} - 1\right)\frac{1}{2}\left(A\varepsilon^j + \frac{\varepsilon^j}{a} - 1\right) - p_3^k p_2^i p_1^{r+m_1+m_2+m_3+m_{12}+j}\right)\left(\frac{1}{\varepsilon^j} - \frac{1}{\gamma^k}\right)$$

$$\ge \sum_{1\le i<k\le r} \sum_{1\le j\le m_{13}} \left(\frac{1}{2}\left(\gamma^k B + p_3^k\right)\frac{1}{2}\left(\gamma^i A + p_2^i\right)\frac{1}{2}\left(A\varepsilon^j + \frac{\varepsilon^j}{a} - 1\right) - p_3^k p_2^i p_1^{r+m_1+m_2+m_3+m_{12}+j}\right)\left(\frac{1}{\varepsilon^j} - \frac{1}{\gamma^k}\right)\left(A\varphi o\acute{\upsilon}\ \frac{\gamma^i}{a} - 1 > p_2^i\ \kappa\alpha\iota\ \frac{\gamma^k}{\beta} - 1 > p_3^k\right)$$

$$\ge \sum_{1\le i<k\le r} \sum_{1\le j\le m_{13}} \left(\frac{1}{2}\left(\gamma^k B + p_3^k\right)\frac{1}{2}\left(\gamma^i A + p_2^i\right)\frac{1}{2}\left(A\varepsilon^j + p_1^j\right) - p_3^k p_2^i p_1^j\right)\left(\frac{1}{\varepsilon^j} - \frac{1}{\gamma^k}\right)\left(A\varphi o\acute{\upsilon}\ \frac{\gamma^i}{a} - 1 > p_2^i\ \kappa\alpha\iota\ \frac{\gamma^k}{\beta} - 1 > p_3^k\right)$$

$$= \sum_{1\le i<k\le r} \sum_{1\le j\le m_{13}} \left(\frac{1}{8}\left(\gamma^k B\gamma^i A + \gamma^k B p_2^i + p_3^k \gamma^i A + p_3^k p_2^i\right)\left(A\varepsilon^j + p_1^j\right) - p_3^k p_2^i p_1^j\right)\left(\frac{1}{\varepsilon^j} - \frac{1}{\gamma^k}\right)$$

$$= \sum_{1\le i<k\le r} \sum_{1\le j\le m_{13}} \left(\frac{1}{8}\left(\gamma^k B\gamma^i A A\varepsilon^j + \gamma^k B\gamma^i A p_1^j + \gamma^k B p_2^i A\varepsilon^j + \gamma^k B p_2^i p_1^j + p_3^k\gamma^i A A\varepsilon^j + p_3^k\gamma^i A p_1^j + p_3^k p_2^i A\varepsilon^j - 7p_3^k p_2^i p_1^j\right)\right)\left(\frac{1}{\varepsilon^j} - \frac{1}{\gamma^k}\right)$$

$$\ge \sum_{1\le i<k\le r} \sum_{1\le j\le m_{13}} \left(\frac{1}{8}\left(A^3\gamma^k\gamma^i\varepsilon^j + A^2\gamma^k\gamma^i p_1^j + A^2\gamma^k p_2^i\varepsilon^j + A\gamma^k p_2^i p_1^j + A^2 p_3^k\gamma^i\varepsilon^j + Ap_3^k\gamma^i p_1^j + Ap_3^k p_2^i\varepsilon^j - 7p_3^k p_2^i p_1^j\right)\right)\left(\frac{1}{\varepsilon^j} - \frac{1}{\gamma^k}\right)(A\varphi o\acute{\upsilon}\ B \ge A)$$

$$= \sum_{1\le i<k\le r}\sum_{1\le j\le m_{13}}\left(\frac{1}{8}\left(A^3\gamma^k\gamma^i\varepsilon^j + A^2\gamma^k\gamma^i\left(\frac{\varepsilon^j}{\alpha}-1\right)+A^2\gamma^k\varepsilon^j\left(\frac{\gamma^i}{2a}-\frac{\gamma^i}{2\beta}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\right.\right.$$

$$+A\gamma^k\left(\frac{\gamma^i}{2a}-\frac{\gamma^i}{2\beta}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\left(\frac{\varepsilon^j}{\alpha}-1\right)+A^2\gamma^i\varepsilon^j\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)$$

$$+A\gamma^i\left(\frac{\varepsilon^j}{\alpha}-1\right)\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)$$

$$+A\varepsilon^j\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\gamma^i}{2a}-\frac{\gamma^i}{2\beta}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)$$

$$\left.\left.-7\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\gamma^i}{2a}-\frac{\gamma^i}{2\beta}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\left(\frac{\varepsilon^j}{\alpha}-1\right)\right)\right)\left(\frac{1}{\varepsilon^j}-\frac{1}{\gamma^k}\right)$$

$$= \sum_{1\le i\le r}\sum_{1\le j\le m_{13}}\left(\frac{1}{8}\left(A^3\gamma^i\varepsilon^j + A^2\gamma^i\left(\frac{\varepsilon^j}{\alpha}-1\right)\right.\right.$$

$$+\left(\frac{\gamma^i}{2\beta}-\frac{\gamma^i}{2\alpha}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\left(2A^2\varepsilon^j+2A\left(\frac{\varepsilon^j}{\alpha}-1\right)+A\left(\frac{\gamma^i}{2\beta}-\frac{\gamma^i}{2\alpha}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\right.$$

$$\left.\left.\left.-\frac{7}{\varepsilon^j}\left(\frac{\gamma^i}{2a}-\frac{\gamma^i}{2\beta}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\left(\frac{\varepsilon^j}{\alpha}-1\right)\right)\right)\frac{1}{\varepsilon^j}S(j)\right)$$

41. <u>Απλοποίηση της Δ_{41}</u>

$$\Delta 41 = \sum_{1\le k\le r} f^k \sum_{1\le i\le r} f^i \sum_{1\le j\le r} f^j\left(\frac{1}{\gamma^j}-\frac{1}{\gamma^k}\right)-\sum_{1\le k\le r} p_3^k \sum_{1\le i\le r} p_2^i \sum_{1\le j\le r} p_1^j\left(\frac{1}{\gamma^j}-\frac{1}{\gamma^k}\right)$$

$$= \sum_{1\le k<j<i\le r}\left(f^k f^i f^j - p_3^k p_2^i p_1^j\right)\left(\frac{1}{\gamma^j}-\frac{1}{\gamma^k}\right)$$

$$= \frac{1}{8}\sum_{1\le i<k<j\le r}\left(\left(\gamma^k B+\frac{\gamma^k}{\beta}-1\right)\left(\gamma^i A+\frac{\gamma^i}{a}-1\right)\left(\gamma^j B+\frac{\gamma^j}{\beta}-1\right)\right.$$

$$\left.-\left(\frac{\gamma^k}{\beta}-\frac{\gamma^k}{\alpha}+\frac{\gamma^k}{\gamma}-1\right)\left(\frac{\gamma^i}{a}-\frac{\gamma^i}{\beta}+\frac{\gamma^i}{\gamma}-1\right)\left(\frac{\gamma^j}{\beta}+\frac{\gamma^j}{\alpha}-\frac{\gamma^j}{\gamma}-1\right)\right)\left(\frac{1}{\gamma^j}-\frac{1}{\gamma^k}\right)$$

$$= \frac{1}{8} \sum_{1 \le i < k < j \le r} \left(\left(\gamma^k B + \frac{\gamma^k}{\beta} - 1 \right) \left(\gamma^i A + \frac{\gamma^i}{a} - 1 \right) \left(\gamma^j B + \frac{\gamma^j}{\beta} - 1 \right) \right.$$
$$\left. - \left(\frac{\gamma^k}{\beta} - \frac{\gamma^k}{\alpha} + \frac{\gamma^k}{\gamma} - 1 \right) \left(\frac{\gamma^i}{a} - \frac{\gamma^i}{\beta} + \frac{\gamma^i}{\gamma} - 1 \right) \left(\frac{\gamma^j}{\beta} + \frac{\gamma^j}{\alpha} - \frac{\gamma^j}{\gamma} - 1 \right) \right) \left(\frac{1}{\gamma^j} - \frac{1}{\gamma^k} \right)$$

$$\ge \frac{1}{8} \sum_{1 \le i < k < j \le r} \left(\left(\gamma^k B + \frac{\gamma^k}{\beta} - 1 \right) \left(\gamma^i A + \frac{\gamma^i}{a} - 1 \right) \left(\gamma^j B + \frac{\gamma^j}{\beta} - 1 \right) \right.$$
$$\left. - \left(\frac{\gamma^k}{\beta} - 1 \right) \left(\frac{\gamma^i}{a} - 1 \right) \left(\frac{\gamma^j}{\beta} - 1 \right) \right) \left(\frac{1}{\gamma^j} - \frac{1}{\gamma^k} \right) (A\varphi o\acute{\upsilon}\ \gamma \ge \beta \ge \alpha)$$

42. <u>Απλοποίηση της Δ_{42}</u>

$$\Delta_{42} = \sum_{1 \le k \le r} f^k \sum_{1 \le i \le m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1 \le j \le m_1} f^{r+j} \left(\frac{1}{\gamma} - \frac{1}{\gamma^k} \right)$$
$$- \sum_{1 \le k \le r} p_3^k \sum_{1 \le i \le m_{12}} p_2^{r+m_1+m_2+m_3+i} \sum_{1 \le j \le m_1} p_1^{r+j} \left(\frac{1}{\gamma} - \frac{1}{\gamma^\kappa} \right)$$

$$= \sum_{1 \le k \le r} \sum_{1 \le i \le m_{12}} \sum_{1 \le j \le m_1} \left(f^k f^{r+m_1+m_2+m_3+i} f^{r+j} - p_3^k p_2^{r+m_1+m_2+m_3+i} p_1^{r+j} \right) \left(\frac{1}{\gamma} - \frac{1}{\gamma^k} \right)$$

$$= \sum_{1 \le k \le r} \sum_{1 \le i \le m_{12}} \left(f^k f^{r+m_1+m_2+m_3+i} m_1 \gamma G \right.$$
$$\left. - p_3^k p_2^{r+m_1+m_2+m_3+i} \left(1 - \frac{\Gamma - r\beta}{2\beta} - \frac{\Gamma}{2\alpha} + \frac{\Gamma}{2\gamma} - \frac{Z - m_{12}\beta}{\beta} - \frac{E - m_{13}\alpha}{\alpha} \right) \right) \left(\frac{1}{\gamma} - \frac{1}{\gamma^k} \right)$$

$$\ge \left(\frac{m_1 \gamma G}{4} - \left(1 - \frac{\Gamma - r\beta}{2\beta} - \frac{\Gamma}{2\alpha} + \frac{\Gamma}{2\gamma} - \frac{Z - m_{12}\beta}{\beta} \right. \right.$$
$$\left. \left. - \frac{E - m_{13}\alpha}{\alpha} \right) \right) \frac{1}{\beta\gamma^2} \sum_{1 \le k \le r} \sum_{1 \le i \le m_{12}} \left((\gamma^k - \beta)(\zeta^i - \gamma) \left(\frac{\gamma^k - \gamma}{\gamma^k} \right) \right)$$

43. <u>Απλοποίηση της Δ_{43}</u>

$$\Delta_{43} = \sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_1} f^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\varepsilon^k}\right)$$
$$-\sum_{1\le k\le m_{13}} p_3^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{23}} p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_1} p_1^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\varepsilon^k}\right)$$

$$= \sum_{1\le k\le m_{13}} \sum_{1\le i\le m_{23}} \sum_{1\le j\le m_1} \left(\frac{1}{2}\left(G\varepsilon^k + p_3^{r+m_1+m_2+m_3+m_{12}+k}\right)\frac{1}{2}\left(B\delta^i + p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i}\right)m_1\gamma G\right.$$
$$- p_3^{r+m_1+m_2+m_3+m_{12}+k} p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i}\left(1-\frac{\Gamma-r\beta}{2\beta}-\frac{\Gamma}{2\alpha}+\frac{\Gamma}{2\gamma}-\frac{Z-m_{12}\beta}{\beta}\right.$$
$$\left.\left.-\frac{E-m_{13}\alpha}{\alpha}\right)\right)\left(\frac{1}{\gamma}-\frac{1}{\varepsilon^k}\right)$$

$$\ge \left(\frac{m_1\gamma G}{4}-\left(1-\frac{\Gamma-r\beta}{2\beta}-\frac{\Gamma}{2\alpha}+\frac{\Gamma}{2\gamma}-\frac{Z-m_{12}\beta}{\beta}\right.\right.$$
$$\left.\left.-\frac{E-m_{13}\alpha}{\alpha}\right)\right)\frac{1}{\alpha\gamma^2}\sum_{1\le k\le m_{13}}\sum_{1\le i\le m_{23}}\left(\frac{\left(\varepsilon^k-\gamma\right)^2}{\varepsilon^k}\left(\delta^i-\alpha\right)\right)$$

44. <u>Απλοποίηση της Δ_{44}</u>

$$\Delta_{44} = \sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_1} f^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\delta^k}\right)$$
$$-\sum_{1\le k\le m_{23}} p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_{23}} p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_1} p_1^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\delta^k}\right)$$

$$\ge \left(\frac{m_1\gamma G}{4}-\left(1-\frac{\Gamma-r\beta}{2\beta}-\frac{\Gamma}{2\alpha}+\frac{\Gamma}{2\gamma}-\frac{Z-m_{12}\beta}{\beta}-\frac{E-m_{13}\alpha}{\alpha}\right)\right)\frac{1}{\alpha\beta\gamma}\sum_{1\le i<k\le m_{23}}\left(\left(\delta^k-\beta\right)\left(\delta^i-\alpha\right)\left(\frac{\delta^k-\gamma}{\delta^k}\right)\right)$$

45. <u>Απλοποίηση της Δ_{45}</u>

$$\Delta_{45} = \sum_{1\le k\le r} f^k \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_1} f^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\gamma^k}\right)$$

$$- \sum_{1\le k\le r} p_3^k \sum_{1\le i\le m_{23}} p_2^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_1} p_1^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\gamma^k}\right)$$

$$\ge \left(\frac{m_1\gamma G}{4} - \left(1 - \frac{\Gamma - r\beta}{2\beta} - \frac{\Gamma}{2\alpha} + \frac{\Gamma}{2\gamma} - \frac{Z - m_{12}\beta}{\beta} - \frac{E - m_{13}\alpha}{\alpha}\right)\right)\frac{1}{\alpha\beta\gamma}\sum_{1\le k\le r}\sum_{1\le i\le m_{23}}\left((\gamma^k-\beta)(\delta^i-\alpha)\left(\frac{\gamma^k-\gamma}{\gamma^k}\right)\right)$$

46. <u>Απλοποίηση της Δ_{46}</u>

$$\Delta_{46} = \sum_{1\le k\le r} f^k \sum_{1\le i\le r} f^i \sum_{1\le j\le m_1} f^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\gamma^k}\right) - \sum_{1\le k\le r} p_3^k \sum_{1\le i\le r} p_2^i \sum_{1\le j\le m_1} p_1^{r+j}\left(\frac{1}{\gamma}-\frac{1}{\gamma^k}\right)$$

$$\ge \left(\frac{m_1\gamma G}{4} - \left(\frac{(r+2)\beta - \Gamma}{2\beta} - \frac{Z - m_{12}\beta}{\beta} - \frac{E - m_{13}\alpha}{\alpha} + \frac{\Gamma(\alpha-\gamma)}{2\beta}\right)\right)\sum_{1\le i<k\le r}\left(\left(\frac{\gamma^k}{\beta}-1\right)\left(\frac{\gamma^i}{a}-1\right)\right)\left(\frac{1}{\gamma}-\frac{1}{\gamma^k}\right)$$

$$= \left(\frac{m_1\gamma G}{4} - \left(\frac{(r+2)\beta - \Gamma}{2\beta} - \frac{Z - m_{12}\beta}{\beta} - \frac{E - m_{13}\alpha}{\alpha} + \frac{\Gamma(\alpha-\gamma)}{2\beta}\right)\right)\frac{1}{\alpha\beta\gamma}\sum_{1\le i<k\le r}\left((\gamma^k-\beta)(\gamma^i-\alpha)\left(\frac{\gamma^k-\gamma}{\gamma^k}\right)\right)$$

47. Απλοποίηση της Δ_{47}

$$\Delta_{47} = \sum_{1\le k\le r} f^k \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j}\left(\frac{1}{\zeta^j}-\frac{1}{\gamma^k}\right)$$
$$- \sum_{1\le k\le r} p_3^k \sum_{1\le i\le m_{12}} p_2^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_{12}} p_1^{r+m_1+m_2+m_3+j}\left(\frac{1}{\zeta^j}-\frac{1}{\gamma^k}\right)$$

$$= \sum_{1\le k\le r}\sum_{1\le i\le m_{12}}\sum_{1\le j\le m_{12}} \Big(f^k f^{r+m_1+m_2+m_3+i} f^{r+m_1+m_2+m_3+j}$$
$$- p_3^k p_2^{r+m_1+m_2+m_3+i} p_1^{r+m_1+m_2+m_3+j}\Big)\left(\frac{1}{\zeta^j}-\frac{1}{\gamma^k}\right)$$

$$\ge \sum_{1\le k\le r}\sum_{1\le i<j\le m_{12}}\Bigg(\frac{1}{8}\Big(\gamma^k BG\zeta^i B\zeta^j + \gamma^k BG\zeta^i p_1^j + \gamma^k B p_2^i B\zeta^j + \gamma^k B p_2^i p_1^j + G\zeta^i p_3^k B\zeta^j$$
$$+ G\zeta^i p_3^k p_1^j + B\zeta^j p_3^k p_2^i - 7 p_3^k p_2^i p_1^j\Big)\Bigg)\left(\frac{1}{\zeta^j}-\frac{1}{\gamma^k}\right)$$

$$= \sum_{1\le k\le r}\sum_{1\le j<i\le m_{12}}\Bigg(\frac{1}{8}\Bigg(\gamma^k B^2 G\zeta^i\zeta^j + \gamma^k BG\zeta^i\left(\frac{\zeta^j}{\beta}-1\right) + \gamma^k B^2\zeta^j\left(\frac{\zeta^i}{\gamma}-1\right)$$
$$+ \gamma^k B\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{\zeta^j}{\beta}-1\right) + G\zeta^i\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)B\zeta^j$$
$$+ G\zeta^i\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\zeta^j}{\beta}-1\right) + B\zeta^j\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\zeta^i}{\gamma}-1\right)$$
$$- 7\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{\zeta^j}{\beta}-1\right)\Bigg)\Bigg)\left(\frac{1}{\zeta^j}-\frac{1}{\gamma^k}\right)$$

$$\ge \sum_{1\le k\le r}\sum_{1\le j<i\le m_{12}}\Bigg(\frac{1}{8}\Bigg(\gamma^k B^3\zeta^i\zeta^j + \gamma^k B^2\zeta^i\left(\frac{\zeta^j}{\beta}-1\right) + \gamma^k B^2\zeta^j\left(\frac{\zeta^i}{\gamma}-1\right)$$
$$+ \gamma^k B\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{\zeta^j}{\beta}-1\right) + B^2\zeta^i\zeta^j\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)$$
$$+ B\zeta^i\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\zeta^j}{\beta}-1\right) + B\zeta^j\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\zeta^i}{\gamma}-1\right)$$
$$- 7\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{\zeta^j}{\beta}-1\right)\Bigg)\Bigg)\left(\frac{1}{\zeta^j}-\frac{1}{\gamma^k}\right)$$

$$= \frac{1}{8} \sum_{1 \le k \le r} \sum_{1 \le j < i \le m_{12}} \Bigg(\Bigg(\gamma^k B^3 \zeta^i \zeta^j + \gamma^k B^2 \zeta^i \left(\frac{\zeta^j}{\beta} - 1 \right) + \gamma^k B^2 \zeta^j \left(\frac{\zeta^i}{\gamma} - 1 \right)$$

$$+ \gamma^k B \left(\frac{\zeta^i}{\gamma} - 1 \right) \left(\frac{\zeta^j}{\beta} - 1 \right)$$

$$+ \left(\frac{\gamma^k}{2\beta} - \frac{\gamma^k}{2\alpha} + \frac{\gamma^k}{2\gamma} - \frac{1}{2} \right) \Bigg(B^2 \zeta^i \zeta^j + B \zeta^i \left(\frac{\zeta^j}{\beta} - 1 \right) + B \zeta^j \left(\frac{\zeta^i}{\gamma} - 1 \right)$$

$$- 7 \left(\frac{\zeta^i}{\gamma} - 1 \right) \left(\frac{\zeta^j}{\beta} - 1 \right) \Bigg) \Bigg) \Bigg) \left(\frac{1}{\zeta^j} - \frac{1}{\gamma^k} \right)$$

48. Απλοποίηση της Δ_{48}

$$\Delta_{48} = \sum_{1 \le k \le r} f^k \sum_{1 \le i \le r} f^i \sum_{1 \le j \le m_{12}} f^{r+m_1+m_2+m_3+j} \left(\frac{1}{\zeta^j} - \frac{1}{\gamma^k} \right)$$

$$- \sum_{1 \le k \le r} p_3^k \sum_{1 \le i \le r} p_2^i \sum_{1 \le j \le m_{12}} p_1^{r+m_1+m_2+m_3+j} \left(\frac{1}{\zeta^j} - \frac{1}{\gamma^k} \right)$$

$$= \sum_{1 \le k \le r} \sum_{1 \le i \le r} \sum_{1 \le j \le m_{12}} \left(f^k f^i f^{r+m_1+m_2+m_3+j} - p_3^k p_2^i p_1^{r+m_1+m_2+m_3+j} \right) \left(\frac{1}{\zeta^j} - \frac{1}{\gamma^k} \right)$$

$$\ge \sum_{1 \le i < k \le r} \sum_{1 \le j \le m_{12}} \Bigg(\frac{1}{2} \left(\gamma^k B + p_3^k \right) \frac{1}{2} \left(\gamma^i A + p_2^i \right) \frac{1}{2} \left(B \zeta^j + p_1^{r+m_1+m_2+m_3+j} \right)$$

$$- p_3^k p_2^i p_1^{r+m_1+m_2+m_3+j} \Bigg) \left(\frac{1}{\zeta^j} - \frac{1}{\gamma^k} \right)$$

$$= \frac{1}{8} \sum_{1 \le i < k \le r} \sum_{1 \le j \le m_{12}} \left(\left(\gamma^k B + p_3^k \right) \left(\gamma^i A + p_2^i \right) \left(B \zeta^j + p_1^j \right) - p_3^k p_2^i p_1^j \right) \left(\frac{1}{\zeta^j} - \frac{1}{\gamma^k} \right)$$

$$= \frac{1}{8} \sum_{1 \le i < k \le r} \sum_{1 \le j \le m_{12}} \Big(\big(\gamma^k B \gamma^i A B \zeta^j + \gamma^k B \gamma^i A p_1^j + \gamma^k B p_2^i B \zeta^j + \gamma^k B p_2^i p_1^j$$

$$+ p_3^k \gamma^i A B \zeta^j + p_3^k \gamma^i A p_1^j + p_3^k p_2^i B \zeta^j - 7 p_3^k p_2^i p_1^j \big) \Big) \left(\frac{1}{\zeta^j} - \frac{1}{\gamma^k} \right)$$

49. Απλοποίηση της Δ_{49}

$$\Delta_{49} = \sum_{1\le k\le r} f^k \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\varepsilon^j}-\frac{1}{\gamma^k}\right)$$

$$-\sum_{1\le k\le r} p_3^k \sum_{1\le i\le m_{12}} p_2^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_{13}} p_1^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\varepsilon^j}-\frac{1}{\gamma^k}\right)$$

$$=\sum_{1\le k\le r}\sum_{1\le i\le m_{12}}\sum_{1\le j\le m_{13}}\left(\frac{1}{2}\left(\gamma^k B+p_3^k\right)\frac{1}{2}\left(G\zeta^i+p_2^{r+m_1+m_2+m_3+i}\right)\frac{1}{2}\left(A\varepsilon^j+p_1^{r+m_1+m_2+m_3+m_{12}+j}\right)\right.$$
$$\left.-\,p_3^k p_2^{r+m_1+m_2+m_3+i} p_1^{r+m_1+m_2+m_3+m_{12}+j}\right)\left(\frac{1}{\varepsilon^j}-\frac{1}{\gamma^k}\right)$$

$$=\frac{1}{8}\sum_{1\le k\le r}\sum_{1\le i\le m_{12}}\sum_{1\le j\le m_{13}}\left(\left(\gamma^k BG\zeta^i A\varepsilon^j+\gamma^k BG\zeta^i p_1^j+\gamma^k Bp_2^i A\varepsilon^j+\gamma^k Bp_2^i p_1^j+p_3^k G\zeta^i A\varepsilon^j+p_3^k G\zeta^i p_1^j\right.\right.$$
$$\left.\left.+\,p_3^k p_2^i A\varepsilon^j-7p_3^k p_2^i p_1^j\right)\right)\left(\frac{1}{\varepsilon^j}-\frac{1}{\gamma^k}\right)$$

50. <u>Απλοποίηση της Δ_{50}</u>

$$\Delta_{50} = \sum_{1\le k\le r} f^k \sum_{1\le i\le r} f^i \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\varepsilon^j}-\frac{1}{\gamma^k}\right)$$

$$-\sum_{1\le k\le r} p_3^k \sum_{1\le i\le r} p_2^i \sum_{1\le j\le m_{13}} p_1^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\varepsilon^j}-\frac{1}{\gamma^k}\right)$$

$$\ge\sum_{1\le k\le r}\sum_{1\le i\le r}\sum_{1\le j\le m_{13}}\left(\frac{1}{2}\left(\gamma^k B+p_3^k\right)\frac{1}{2}\left(\gamma^i A+p_2^i\right)\frac{1}{2}\left(A\varepsilon^j+p_1^{r+m_1+m_2+m_3+m_{12}+j}\right)\right.$$
$$\left.-\,p_3^k p_2^i p_1^{r+m_1+m_2+m_3+m_{12}+j}\right)\left(\frac{1}{\varepsilon^j}-\frac{1}{\gamma^k}\right)$$

$$=\sum_{1\le k\le r}\sum_{1\le i\le r}\sum_{1\le j\le m_{13}}\left(\frac{1}{8}\left(\gamma^k\gamma^i AB+\gamma^k Bp_2^i+p_3^k\gamma^i A+p_3^k p_2^i\right)\left(A\varepsilon^j+p_1^{r+m_1+m_2+m_3+m_{12}+j}\right)\right.$$
$$\left.-\,p_3^k p_2^i p_1^{r+m_1+m_2+m_3+m_{12}+j}\right)\left(\frac{1}{\varepsilon^j}-\frac{1}{\gamma^k}\right)$$

$$\ge\sum_{1\le k\le r}\sum_{1\le i\le r}\sum_{1\le j\le m_{13}}\left(\frac{1}{8}\left(\gamma^k Bp_2^i+p_3^k\gamma^i A+p_3^k p_2^i\right)\left(A\varepsilon^j+p_1^j\right)-p_3^k p_2^i p_1^j\right)\left(\frac{1}{\varepsilon^j}-\frac{1}{\gamma^k}\right)$$

$$= \sum_{1\le k<i\le r} \sum_{1\le j\le m_{13}} \Big(\frac{1}{8}\big(\gamma^k B p_2^i A\varepsilon^j + \gamma^k B p_2^i p_1^j + p_3^k \gamma^i A A\varepsilon^j + p_3^k \gamma^i A p_1^j + A\varepsilon^j p_3^k p_2^i + p_3^k p_2^i p_1^j\big) - p_3^k p_2^i p_1^j\Big)\Big(\frac{1}{\varepsilon^j} - \frac{1}{\gamma^k}\Big)$$

51. <u>Απλοποίηση της Δ_{51}</u>

$$\Delta 51 = \sum_{1\le k\le r} f^k \sum_{1\le i\le r} f^i \sum_{1\le j\le r} f^j \Big(\frac{1}{\gamma^j} - \frac{1}{\gamma^k}\Big) - \sum_{1\le k\le r} p_3^k \sum_{1\le i\le r} p_2^i \sum_{1\le j\le r} p_1^j \Big(\frac{1}{\gamma^j} - \frac{1}{\gamma^k}\Big)$$

$$= \sum_{1\le k<j<i\le r} \big(f^k f^i f^j - p_3^k p_2^i p_1^j\big)\Big(\frac{1}{\gamma^j} - \frac{1}{\gamma^k}\Big)$$

$$= \frac{1}{8} \sum_{1\le k<i<j\le r} \Bigg(\Big(\gamma^k B + \frac{\gamma^k}{\beta} - 1\Big)\Big(\gamma^i A + \frac{\gamma^i}{a} - 1\Big)\Big(\gamma^j B + \frac{\gamma^j}{\beta} - 1\Big) - \Big(\frac{\gamma^k}{\beta} - \frac{\gamma^k}{\alpha} + \frac{\gamma^k}{\gamma} - 1\Big)\Big(\frac{\gamma^i}{a} - \frac{\gamma^i}{\beta} + \frac{\gamma^i}{\gamma} - 1\Big)\Big(\frac{\gamma^j}{\beta} + \frac{\gamma^j}{\alpha} - \frac{\gamma^j}{\gamma} - 1\Big)\Bigg)\Big(\frac{1}{\gamma^j} - \frac{1}{\gamma^k}\Big)$$

$$= \frac{1}{8} \sum_{1\le k<i<j\le r} \Bigg(\Big(\gamma^k B + \frac{\gamma^k}{\beta} - 1\Big)\Big(\gamma^i A + \frac{\gamma^i}{a} - 1\Big)\Big(\gamma^j B + \frac{\gamma^j}{\beta} - 1\Big) - \Big(\frac{\gamma^k}{\beta} - \frac{\gamma^k}{\alpha} + \frac{\gamma^k}{\gamma} - 1\Big)\Big(\frac{\gamma^i}{a} - \frac{\gamma^i}{\beta} + \frac{\gamma^i}{\gamma} - 1\Big)\Big(\frac{\gamma^j}{\beta} + \frac{\gamma^j}{\alpha} - \frac{\gamma^j}{\gamma} - 1\Big)\Bigg)\Big(\frac{1}{\gamma^j} - \frac{1}{\gamma^k}\Big)$$

$$\ge \frac{1}{8} \sum_{1\le k<i<j\le r} \Bigg(\Big(\gamma^k B + \frac{\gamma^k}{\beta} - 1\Big)\Big(\gamma^i A + \frac{\gamma^i}{a} - 1\Big)\Big(\gamma^j B + \frac{\gamma^j}{\beta} - 1\Big) - \Big(\frac{\gamma^k}{\beta} - 1\Big)\Big(\frac{\gamma^i}{a} - 1\Big)\Big(\frac{\gamma^j}{\beta} - 1\Big)\Bigg)\Big(\frac{1}{\gamma^j} - \frac{1}{\gamma^k}\Big) \text{(Αφού } \gamma \ge \beta \ge \alpha\text{)}$$

52. <u>Απλοποίηση της Δ_{52}</u>

$$\Delta_{52} = \sum_{1\le k\le m_3} f^{r+m_1+m_2+k} \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} f^{r+m_1+m_2+m_3+i}\left(\frac{2}{\zeta^i}-\frac{1}{a}\right)$$
$$-\sum_{1\le k\le m_3} p_3^{r+m_1+m_2+k} \sum_{1\le i\le m_{12}} p_2^{r+m_1+m_2+m_3+i} p_1^{r+m_1+m_2+m_3+i}\left(\frac{2}{\zeta^i}-\frac{1}{a}\right)$$

$$= \sum_{1\le k\le m_3}\sum_{1\le i\le m_{12}} \Big(f^{r+m_1+m_2+k} f^{r+m_1+m_2+m_3+i} f^{r+m_1+m_2+m_3+i}$$
$$- p_3^{r+m_1+m_2+k} p_2^{r+m_1+m_2+m_3+i} p_1^{r+m_1+m_2+m_3+i}\Big)\left(\frac{2}{\zeta^i}-\frac{1}{a}\right)$$

$$= \sum_{1\le i\le m_{12}} \Bigg(m_3 aA\frac{1}{2}\left(B\zeta^i + p_2^{r+m_1+m_2+m_3+i}\right)\frac{1}{2}\left(G\zeta^i + p_1^{r+m_1+m_2+m_3+i}\right)$$
$$-\left(1-\frac{\Gamma-r\beta}{2\beta}+\frac{\Gamma}{2\alpha}-\frac{\Gamma}{2\gamma}-\frac{E-m_{13}\gamma}{\gamma}-\frac{\Delta-m_{23}\beta}{\beta}\right) p_2^{r+m_1+m_2+m_3+i} p_1^{r+m_1+m_2+m_3+i}\Bigg)\left(\frac{2}{\zeta^i}-\frac{1}{a}\right)$$
$$= \sum_{1\le i\le m_{12}} \Bigg(m_3 aA\frac{1}{2}\left(B\zeta^i + p_2^{r+m_1+m_2+m_3+i}\right)\frac{1}{2}\left(G\zeta^i + p_1^{r+m_1+m_2+m_3+i}\right)$$
$$-\left(1-\frac{\Gamma-r\beta}{2\beta}+\frac{\Gamma}{2\alpha}-\frac{\Gamma}{2\gamma}-\frac{E-m_{13}\gamma}{\gamma}-\frac{\Delta-m_{23}\beta}{\beta}\right) p_2^{r+m_1+m_2+m_3+i} p_1^{r+m_1+m_2+m_3+i}\Bigg)\left(\frac{2}{\zeta^i}-\frac{1}{a}\right)$$

$$\ge \Bigg(\frac{m_3 aA}{4}-\Bigg(1-\frac{\Gamma-r\beta}{2\beta}+\frac{\Gamma}{2\alpha}-\frac{\Gamma}{2\gamma}-\frac{E-m_{13}\gamma}{\gamma}$$
$$-\frac{\Delta-m_{23}\beta}{\beta}\Bigg)\Bigg)\frac{1}{\alpha\beta\gamma}\sum_{1\le i\le m_{12}}\left((\zeta^i-\gamma)(\zeta^i-\beta)\left(\frac{2\alpha-\zeta^i}{\zeta^i}\right)\right)$$

53. <u>Απλοποίηση της Δ_{53}</u>

$$\Delta_{53} = \sum_{1\le k\le m_3} f^{r+m_1+m_2+k} \sum_{1\le i\le r} f^i f^i\left(\frac{2}{\gamma^i}-\frac{1}{a}\right)$$
$$-\sum_{1\le k\le m_3} p_3^{r+m_1+m_2+k} \sum_{1\le i\le r} p_2^i p_1^i\left(\frac{2}{\gamma^i}-\frac{1}{a}\right)$$

$$= \sum_{1\le k\le m_3} \sum_{1\le i\le r} \left(\frac{m_3 aA}{4}\left(\gamma^i A + \frac{\gamma^i}{a} - 1\right)\left(\gamma^i B + \frac{\gamma^i}{\beta} - 1\right) - \left(1 - \frac{\Gamma - r\beta}{2\beta} + \frac{\Gamma}{2\alpha} - \frac{\Gamma}{2\gamma} - \frac{E - m_{13}\gamma}{\gamma} - \frac{\Delta - m_{23}\beta}{\beta}\right) p_2^i p_1^i\right)\left(\frac{2}{\gamma^i} - \frac{1}{a}\right)$$

$$\ge \left(\frac{m_3 aA}{4} - \left(1 - \frac{\Gamma - r\beta}{2\beta} + \frac{\Gamma}{2\alpha} - \frac{\Gamma}{2\gamma} - \frac{E - m_{13}\gamma}{\gamma} - \frac{\Delta - m_{23}\beta}{\beta}\right)\right) \sum_{1\le i\le r}\left(\left(\frac{\gamma^i}{a} - 1\right)\left(\frac{\gamma^i}{\beta} - 1\right)\left(\frac{2}{\gamma^i} - \frac{1}{a}\right)\right)$$

$$= \left(\frac{m_3 aA}{4} - \left(1 - \frac{\Gamma - r\beta}{2\beta} + \frac{\Gamma}{2\alpha} - \frac{\Gamma}{2\gamma} - \frac{E - m_{13}\gamma}{\gamma} - \frac{\Delta - m_{23}\beta}{\beta}\right)\right) \frac{1}{\alpha^2\beta} \sum_{1\le i\le r}\left((\gamma^i - \alpha)(\gamma^i - \beta)\left(\frac{2\alpha - \gamma^i}{\gamma^i}\right)\right)$$

54. Απλοποίηση της Δ_{54}

$$\Delta_{54} = \sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} f^{r+m_1+m_2+m_3+i}\left(\frac{2}{\zeta^i} - \frac{1}{\varepsilon^k}\right)$$
$$- \sum_{1\le k\le m_{13}} p_3^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{12}} p_2^{r+m_1+m_2+m_3+i} p_1^{r+m_1+m_2+m_3+i}\left(\frac{2}{\zeta^i} - \frac{1}{\varepsilon^k}\right)$$

$$\sum_{1\le k\le m_{13}} \sum_{1\le i\le m_{12}} \left(\frac{1}{2}\left(G\varepsilon^k + p_3^{r+m_1+m_2+m_3+m_{12}+k}\right)\frac{1}{2}\left(G\zeta^i + p_2^{r+m_1+m_2+m_3+i}\right)\frac{1}{2}\left(B\zeta^i + p_1^{r+m_1+m_2+m_3+i}\right) - p_3^{r+m_1+m_2+m_3+m_{12}+k} p_2^{r+m_1+m_2+m_3+i} p_1^{r+m_1+m_2+m_3+i}\right)\left(\frac{2}{\zeta^i} - \frac{1}{\varepsilon^k}\right)$$

$$= \sum_{1\le k\le m_{13}} \sum_{1\le i\le m_{12}} \left(\frac{1}{2}\left(G\varepsilon^k G\zeta^i B\zeta^i + G\varepsilon^k G\zeta^i p_1^i + G\varepsilon^k p_2^i B\zeta^i + G\varepsilon^k p_2^i p_1^i + p_3^k G\zeta^i B\zeta^i + p_3^k G\zeta^i p_1^i + p_3^k p_2^i B\zeta^i + p_3^k p_2^i p_1^i\right) - p_3^k p_2^i p_1^i\right)\left(\frac{2}{\zeta^i} - \frac{1}{\varepsilon^k}\right)$$

$$= \frac{1}{8} \sum_{1\le k\le m_{13}} \sum_{1\le i\le m_{12}} \left(G\varepsilon^k G\zeta^i B\zeta^i + G\varepsilon^k G\zeta^i p_1^i + G\varepsilon^k p_2^i B\zeta^i + G\varepsilon^k p_2^i p_1^i + p_3^k G\zeta^i B\zeta^i + p_3^k G\zeta^i p_1^i + p_3^k p_2^i B\zeta^i - 7 p_3^k p_2^i p_1^i\right)\left(\frac{2}{\zeta^i} - \frac{1}{\varepsilon^k}\right)$$

$$= \frac{1}{8} \sum_{1 \le k \le m_{13}} \sum_{1 \le i \le m_{12}} \left(G\varepsilon^k G\zeta^i B\zeta^i + G\varepsilon^k G\zeta^i \left(\frac{\zeta^i}{\beta} - 1\right) + G\varepsilon^k \left(\frac{\zeta^i}{\gamma} - 1\right) B\zeta^i + G\varepsilon^k \left(\frac{\zeta^i}{\gamma} - 1\right)\left(\frac{\zeta^i}{\beta} - 1\right) \right.$$
$$+ \left(\frac{\varepsilon^k}{\gamma} - 1\right) G\zeta^i B\zeta^i + \left(\frac{\varepsilon^k}{\gamma} - 1\right) G\zeta^i \left(\frac{\zeta^i}{\beta} - 1\right) + \left(\frac{\varepsilon^k}{\gamma} - 1\right)\left(\frac{\zeta^i}{\gamma} - 1\right) B\zeta^i$$
$$\left. - 7\left(\frac{\varepsilon^k}{\gamma} - 1\right)\left(\frac{\zeta^i}{\gamma} - 1\right)\left(\frac{\zeta^i}{\beta} - 1\right)\right)\left(\frac{2}{\zeta^i} - \frac{1}{\varepsilon^k}\right)$$

55. <u>Απλοποίηση της $\varDelta_{55}$</u>

$$\varDelta_{55} = \sum_{1 \le k \le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1 \le i \le r} f^i f^i \left(\frac{2}{\gamma^i} - \frac{1}{\varepsilon^k}\right)$$
$$- \sum_{1 \le k \le m_{13}} p_3^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1 \le i \le r} p_2^i p_1^i \left(\frac{2}{\gamma^i} - \frac{1}{\varepsilon^k}\right)$$

$$= \sum_{1 \le k \le m_{13}} \sum_{1 \le i \le r} \left(\frac{1}{2}\left(G\varepsilon^k + p_3^{r+m_1+m_2+m_3+m_{12}+k}\right)\frac{1}{2}\left(\gamma^i A + \frac{\gamma^i}{a} - 1\right)\frac{1}{2}\left(\gamma^i B + \frac{\gamma^i}{\beta} - 1\right)\right.$$
$$\left. - p_3^{r+m_1+m_2+m_3+m_{12}+k} p_2^i p_1^i\right)\left(\frac{2}{\gamma^i} - \frac{1}{\varepsilon^k}\right)$$

$$\ge \frac{1}{8} \sum_{1 \le k \le m_{13}} \sum_{1 \le i \le r} \left((G\varepsilon^k + p_3^k)(\gamma^i A + p_2^i)(\gamma^i B + p_1^i) - p_3^k p_2^i p_1^i\right)\left(\frac{2}{\gamma^i} - \frac{1}{\varepsilon^k}\right)\left(Aφού\ \frac{\gamma^i}{a} - 1 \right.$$
$$\left. \ge p_2^i\ και\ \frac{\gamma^i}{\beta} - 1 \ge p_1^i\right)$$

$$= \frac{1}{8} \sum_{1 \le k \le m_{13}} \sum_{1 \le i \le r} \left(\left(G\varepsilon^k \gamma^i A \gamma^i B + G\varepsilon^k \gamma^i A p_1^i + G\varepsilon^k p_2^i \gamma^i B + G\varepsilon^k p_2^i p_1^i + p_3^k \gamma^i A \gamma^i B + p_3^k \gamma^i A p_1^i \right.\right.$$
$$\left.\left. + p_3^k p_2^i \gamma^i B - 7 p_3^k p_2^i p_1^i\right)\right)\left(\frac{2}{\gamma^i} - \frac{1}{\varepsilon^k}\right)$$

56. <u>Απλοποίηση της $\varDelta_{56}$</u>

$$\varDelta_{56} = \sum_{1 \le k \le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1 \le i \le m_{12}} f^{r+m_1+m_2+m_3+i} f^{r+m_1+m_2+m_3+i} \left(\frac{2}{\zeta^i} - \frac{1}{\delta^k}\right)$$

$$- \sum_{1\le k\le m_{23}} p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_{12}} p_2^{r+m_1+m_2+m_3+i} p_1^{r+m_1+m_2+m_3+i} \left(\frac{2}{\zeta^i} - \frac{1}{\delta^k}\right)$$

$$= \sum_{1\le k\le m_{23}} \sum_{1\le i\le m_{12}} \left(\frac{1}{2}\left(\delta^k B + p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k}\right) \frac{1}{2}\left(G\zeta^i + p_2^{r+m_1+m_2+m_3+i}\right) \frac{1}{2}\left(B\zeta^i + p_1^{r+m_1+m_2+m_3+i}\right) - p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} p_2^{r+m_1+m_2+m_3+i} p_1^{r+m_1+m_2+m_3+i}\right)\left(\frac{2}{\zeta^i} - \frac{1}{\delta^k}\right)$$

$$= \sum_{1\le k\le m_{23}} \sum_{1\le i\le m_{12}} \left(\frac{1}{8}\left(\delta^k BG\zeta^i B\zeta^i + \delta^k BG\zeta^i p_1^i + \delta^k B p_2^i B\zeta^i + \delta^k B p_2^i p_1^i + p_3^k G\zeta^i B\zeta^i + p_3^k G\zeta^i p_1^i + p_3^k p_2^i B\zeta^i - 7 p_3^k p_2^i p_1^i\right)\right)\left(\frac{2}{\zeta^i} - \frac{1}{\delta^k}\right)$$

57. <u>Απλοποίηση της Δ_{57}</u>

$$\Delta_{57} = \sum_{1\le k\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{23}+k} \sum_{1\le i\le r} f^i f^i \left(\frac{2}{\gamma^i} - \frac{1}{\delta^k}\right) - \sum_{1\le k\le m_{23}} p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\le i\le m_{12}} p_2^i p_1^i \left(\frac{2}{\gamma^i} - \frac{1}{\delta^k}\right)$$

$$= \sum_{1\le k\le m_{23}} \sum_{1\le i\le r} \left(\frac{1}{2}\left(\delta^k B + p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k}\right) \frac{1}{2}\left(\gamma^i A + \frac{\gamma^i}{a} - 1\right) \frac{1}{2}\left(\gamma^i B + \frac{\gamma^i}{\beta} - 1\right) - p_3^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} p_2^i p_1^i\right)\left(\frac{2}{\gamma^i} - \frac{1}{\delta^k}\right)$$

$$\ge \frac{1}{8} \sum_{1\le k\le m_{23}} \sum_{1\le i\le r} \left((\delta^k B + p_3^k)(\gamma^i A + p_2^i)(\gamma^i B + p_1^i) - p_3^k p_2^i p_1^i\right)\left(\frac{2}{\gamma^i} - \frac{1}{\delta^k}\right)\left(A\varphi o\acute{\upsilon}\ \frac{\gamma^i}{a} - 1 \ge p_2^i \ \kappa\alpha\iota\ \frac{\gamma^i}{\beta} - 1 \ge p_1^i\right)$$

$$= \frac{1}{8} \sum_{1\le k\le m_{23}} \sum_{1\le i\le r} \left(\left(\delta^k B\gamma^i A\gamma^i B + \delta^k B\gamma^i A p_1^i + \delta^k B p_2^i \gamma^i B + \delta^k B p_2^i p_1^i + p_3^k \gamma^i A\gamma^i B + p_3^k \gamma^i A p_1^i + p_3^k p_2^i \gamma^i B - 7 p_3^k p_2^i p_1^i\right)\right)\left(\frac{2}{\gamma^i} - \frac{1}{\delta^k}\right)$$

58. <u>Απλοποίηση της Δ_{58}</u>

$$\Delta_{58} = \sum_{1\le k\le r} f^k \sum_{1\le i\le r} f^i f^i\left(\frac{2}{\gamma^i}-\frac{1}{\zeta^k}\right) - \sum_{1\le k\le r} p_3^k \sum_{1\le i\le r} p_2^i p_1^i\left(\frac{2}{\gamma^i}-\frac{1}{\gamma^k}\right)$$

$$= \sum_{1\le i<k\le r}\left(f^k f^i f^i - p_3^k p_2^i p_1^i\right)\left(\frac{2}{\gamma^i}-\frac{1}{\zeta^k}\right)$$

$$= \sum_{1\le i<k\le r}\left(\frac{1}{8}\left(\gamma^k B+\frac{\gamma^k}{\beta}-1\right)\left(\gamma^i A+\frac{\gamma^i}{a}-1\right)\left(\gamma^j B+\frac{\gamma^j}{\beta}-1\right)-p_3^k p_2^i p_1^i\right)\left(\frac{2}{\gamma^i}-\frac{1}{\zeta^k}\right)$$

$$\ge \sum_{1\le i<k\le r}\left(\frac{1}{8}(\gamma^k B+p_3^k)(\gamma^i A+p_2^i)(\gamma^j B+p_1^i)-p_3^k p_2^i p_1^i\right)\left(\frac{2}{\gamma^i}-\frac{1}{\zeta^k}\right)\left(Αφού\ \frac{\gamma^k}{\beta}-1 \ge p_3^k, \frac{\gamma^i}{a}-1\ge p_2^i\ και\ \frac{\gamma^i}{\beta}-1\ge p_1^i\right)$$

59. <u>Απλοποίηση της Δ_{59}</u>

$$\Delta_{59} = \sum_{1\le k\le r} f^k \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} f^{r+m_1+m_2+m_3+i}\left(\frac{2}{\zeta^i}-\frac{1}{\gamma^k}\right)$$

$$-\sum_{1\le k\le r} p_3^k \sum_{1\le i\le m_{12}} p_2^{r+m_1+m_2+m_3+i} p_1^{r+m_1+m_2+m_3+i}\left(\frac{2}{\zeta^i}-\frac{1}{\gamma^k}\right)$$

$$\ge \sum_{1\le k\le r}\sum_{1\le i\le m_{12}}\left(\frac{1}{8}(\gamma^k B+p_3^k)(G\zeta^i+p_2^i)(B\zeta^i+p_1^i)-p_3^k p_2^i p_1^i\right)\left(\frac{2}{\zeta^i}-\frac{1}{\gamma^k}\right)$$

60. <u>Απλοποίηση της Δ_{60}</u>

$$\Delta_{60} = \sum_{1\le k\le r} f^k \sum_{1\le i\le r} f^i f^i\left(\frac{2}{\zeta^i}-\frac{1}{\delta^k}\right) - \sum_{1\le k\le r} p_3^k \sum_{1\le i\le r} p_2^i p_1^i\left(\frac{2}{\zeta^i}-\frac{1}{\delta^k}\right)$$

$$= \sum_{1\le k<i\le r}\left(\frac{1}{8}(\gamma^k B+p_3^k)(\gamma^i A+p_2^i)(\gamma^i B+p_1^i)-p_3^k p_2^i p_1^i\right)\left(\frac{2}{\zeta^i}-\frac{1}{\delta^k}\right)$$

Ακολούθως παίρνουμε μερικές από τις υπόλοιπες διαφορές και τις συνδυάζουμε ανάλογα βλέποντας τους πίνακες (Παράρτημα Α,Β) και παίρνοντας τις περιπτώσεις που έχουμε ίδιους δείκτες (i, k και j). Οι πιο κάτω απλοποιήσεις θα χρησιμοποιηθούν έτσι ώστε να τις προσθέσουμε σε μερικές απο τις πιο πάνω διαφορές όπως θα δούμε στη συνέχεια :

- Απλοποίηση της Δ_{62}και Δ_{65}

$$\Delta_{62} = \sum_{1\leq k\leq m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+k} \sum_{1\leq i\leq m_3} f^{r+m_1+m_2+i} \sum_{1\leq j\leq m_1} f^{r+j}\left(\frac{1}{\alpha}-\frac{1}{\delta^k}\right)$$

$$= m_3 aAm_1\gamma G \sum_{1\leq k\leq m_{23}} \frac{1}{2}\left(B\delta^k+\frac{\delta^k}{\beta}-1\right)\left(\frac{1}{\alpha}-\frac{1}{\delta^k}\right)$$

$$\geq m_1 m_3 aA\gamma G \sum_{1\leq k\leq m_{23}} \frac{1}{2}\left(B\delta^k+\frac{\delta^k}{\beta}-1\right)\left(\frac{1}{\gamma}-\frac{1}{\delta^k}\right) (A\varphi o\acute{\upsilon}\ \gamma\geq\beta\geq\alpha)$$

$$\geq m_1 m_3 aA\gamma G \sum_{1\leq k\leq m_{23}} \frac{1}{2}\left(\frac{\delta^k}{\beta}-1\right)\left(\frac{1}{\gamma}-\frac{1}{\delta^k}\right)$$

$$= \frac{m_1 m_3 aA\gamma G}{2\beta^2} \sum_{1\leq k\leq m_{23}} \left(\frac{\left(\delta^k-\gamma\right)^2}{\delta^k}\right)$$

Με τον ίδιο τρόπο έχουμε,

$$\Delta_{65} = \sum_{1\leq k\leq m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{23}+k} \sum_{1\leq i\leq m_3} f^{r+m_1+m_2+i} \sum_{1\leq j\leq m_2} f^{r+m_1+j}\left(\frac{1}{\alpha}-\frac{1}{\delta^k}\right)$$

$$\geq \frac{m_2 m_3 aA\beta B}{2\beta^2} \sum_{1\leq k\leq m_{23}} \left(\frac{\left(\delta^k-\gamma\right)^2}{\delta^k}\right)$$

Τις δυο αυτές απλοποιημένες διαφορές θα τις προσθέσουμε στην συνέχεια με τη διαφορά Δ_2.

- Απλοποίηση της Δ_{63}και Δ_{66}

$$\Delta_{63} = \sum_{1\leq k\leq m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\leq i\leq m_3} f^{r+m_1+m_2+i} \sum_{1\leq j\leq m_1} f^{r+j}\left(\frac{1}{\alpha}-\frac{1}{\varepsilon^k}\right)$$

$$= m_3 aAm_1\gamma G \sum_{1\leq k\leq m_{13}} \left(\frac{1}{2}\left(G\varepsilon^k-\frac{\varepsilon^k}{\gamma}+1\right)\left(\frac{1}{\alpha}-\frac{1}{\varepsilon^k}\right)\right)$$

$$\geq m_1 m_3 aA\gamma G \sum_{1\leq k\leq m_{13}} \left(\frac{1}{2}\left(G\varepsilon^k + \frac{\varepsilon^k}{\gamma} - 1\right)\left(\frac{1}{\gamma} - \frac{1}{\varepsilon^k}\right)\right) (A\varphi o\acute{\upsilon}\ \gamma \geq \beta \geq \alpha)$$

$$\geq m_1 m_3 aA\gamma G \sum_{1\leq k\leq m_{13}} \left(\frac{1}{2}\left(\frac{\varepsilon^k}{\gamma} - 1\right)\left(\frac{1}{\beta} - \frac{1}{\varepsilon^k}\right)\right)$$

$$= \frac{m_1 m_3 aA\gamma G}{2\beta\gamma} \sum_{1\leq k\leq m_{13}} \left(\frac{(\varepsilon^k - \gamma)^2}{\varepsilon^k}\right)$$

Με τον ίδιο τρόπο έχουμε,

$$\Delta_{66} = \sum_{1\leq k\leq m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\leq i\leq m_3} f^{r+m_1+m_2+i} \sum_{1\leq j\leq m_2} f^{r+m_1+j} \left(\frac{1}{\alpha} - \frac{1}{\varepsilon^k}\right)$$

$$\geq \frac{m_2 m_3 aA\beta B}{2\beta\gamma} \sum_{1\leq k\leq m_{13}} \left(\frac{(\varepsilon^k - \gamma)^2}{\varepsilon^k}\right)$$

Τις δυο αυτές απλοποιημένες διαφορές θα τις προσθέσουμε στην συνέχεια με τη διαφορά Δ_3.

- Απλοποίηση της Δ_{64}και Δ_{67}

$$\Delta_{64} = \sum_{1\leq k\leq r} f^k \sum_{1\leq i\leq m_3} f^{r+m_1+m_2+i} \sum_{1\leq j\leq m_2} f^{r+m_1+j} \left(\frac{1}{\alpha} - \frac{1}{\gamma^k}\right)$$

$$\geq \frac{m_2 m_3 aA\beta B}{2\beta^2} \sum\nolimits_{1\leq k\leq r} \left(\frac{(\gamma^k - \gamma)^2}{\gamma^k}\right)$$

και,

$$\Delta_{67} = \sum_{1\leq k\leq r} f^k \sum_{1\leq i\leq m_3} f^{r+m_1+m_2+i} \sum_{1\leq j\leq m_1} f^{r+j} \left(\frac{1}{\alpha} - \frac{1}{\gamma^k}\right)$$

$$\geq \frac{m_1 m_3 aA\gamma G}{2\beta^2} \sum\nolimits_{1\leq k\leq r} \left(\frac{(\gamma^k - \gamma)^2}{\gamma^k}\right)$$

Τις δυο αυτές απλοποιημένες διαφορές θα τις προσθέσουμε στην συνέχεια με τη διαφορά Δ_4.

- Απλοποίηση της Δ_{68}και Δ_{74}

$$\Delta_{68} = \sum_{1\le k\le r} f^k \sum_{1\le i\le m_3} f^{r+m_1+m_2+i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j}\left(\frac{1}{\alpha}-\frac{1}{\gamma^k}\right)$$

$$= m_3 aA \sum_{1\le k\le r}\sum_{1\le j\le m_{12}} f^k f^{r+m_1+m_2+m_3+j}\left(\frac{1}{\alpha}-\frac{1}{\gamma^k}\right)$$

$$= m_3 aA \sum_{1\le k\le r}\sum_{1\le j\le m_{12}} \left(\frac{1}{2}\left(\gamma^k B+\frac{\gamma^k}{\beta}-1\right)\frac{1}{2}\left(B\zeta^i+\frac{\zeta^i}{\beta}-1\right)\left(\frac{1}{\alpha}-\frac{1}{\gamma^k}\right)\right)$$

$$\ge \frac{m_3 aA}{4\beta^3}\sum_{1\le k\le r}\sum_{1\le j\le m_{12}}\left(\frac{(\gamma^k-\gamma)^2}{\gamma^k}(\zeta^j-\beta)\right)$$

και,

$$\Delta_{74} = \sum_{1\le k\le r} f^k \sum_{1\le i\le m_1} f^{r+i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j}\left(\frac{1}{\gamma}-\frac{1}{\gamma^k}\right)$$

$$\ge \frac{m_1\gamma G}{4\beta^3}\sum_{1\le k\le r}\sum_{1\le j\le m_{12}}\left(\frac{(\gamma^k-\gamma)^2}{\gamma^k}(\zeta^j-\beta)\right)$$

Τις δυο αυτές απλοποιημένες διαφορές θα τις προσθέσουμε στην συνέχεια με τη διαφορά Δ_5.

- Απλοποίηση της Δ_{70}και Δ_{73}

$$\Delta_{70} = \sum_{1\le k\le r} f^k \sum_{1\le i\le m_3} f^{r+m_1+m_2+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\alpha}-\frac{1}{\gamma^k}\right)$$

$$= m_3 aA \sum_{1\le k\le r}\sum_{1\le j\le m_{13}} f^k f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\alpha}-\frac{1}{\gamma^k}\right)$$

$$= m_3 aA \sum_{1\le k\le r}\sum_{1\le j\le m_{13}} \left(\frac{1}{2}\left(\gamma^k B+\frac{\gamma^k}{\beta}-1\right)\frac{1}{2}\left(A\varepsilon^j+\frac{\varepsilon^j}{\alpha}-1\right)\left(\frac{1}{\alpha}-\frac{1}{\gamma^k}\right)\right)$$

$$\ge \frac{m_3 aA}{4\alpha\beta^2}\sum_{1\le k\le r}\sum_{1\le j\le m_{13}}\left(\frac{(\gamma^k-\gamma)^2}{\gamma^k}(\varepsilon^j-a)\right)$$

και,

$$\Delta_{73} = \sum_{1\le k\le r} f^k \sum_{1\le i\le m_1} f^{r+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\gamma}-\frac{1}{\gamma^k}\right)$$

$$\ge \frac{m_1\gamma G}{4\alpha\beta^2}\sum_{1\le k\le r}\sum_{1\le j\le m_{13}}\left(\frac{(\gamma^k-\gamma)^2}{\gamma^k}(\varepsilon^j-a)\right)$$

Τις δυο αυτές απλοποιημένες διαφορές θα τις προσθέσουμε στην συνέχεια με τη διαφορά Δ_6.

8.3 Πρόσθεση των διαφορών

Συνεχίζουμε με τον ίδιο τρόπο όπως πριν, παίρνοντας διαφορές από τις $\Delta_{62},\dots,\Delta_{110}$ και τις απλοποιούμε έτσι ώστε στην συνέχεια να τις προσθέσουμε με μερικές από τις διαφορές $\Delta_1,\Delta_3,\dots,\Delta_{60}$ που δείξαμε πιο πάνω. Έτσι ο συνδυασμός και η πρόσθεση τους μας δίνουν τα ακόλουθα :

A

$$+\frac{m_1m_2\beta B^2\gamma G}{2}\left(\frac{\Delta-m_{23}\beta}{\beta}\right)+\left(\frac{m_1m_2\beta B\gamma G+m_1m_3aA\gamma G+m_2m_3aA\beta B}{2}-\left(\frac{(r+2)\alpha-\Gamma}{2\alpha}-\frac{Z-m_{12}\gamma}{\gamma}-\frac{\Delta-m_{23}\alpha}{\alpha}+\frac{\Gamma(\gamma-\beta)}{2\beta}\right)\left(\frac{(r+2)\beta-\Gamma}{2\beta}-\frac{Z-m_{12}\beta}{\beta}-\frac{E-m_{13}\alpha}{\alpha}+\frac{\Gamma(\alpha-\gamma)}{2\beta}\right)\right)\frac{1}{\beta^2}\sum_{1\le k\le m_{23}}\left(\frac{\left(\delta^k-\beta\right)^2}{\delta^k}\right)$$

$$+\frac{m_1m_2\beta GB\gamma G}{2}\left(\frac{E-m_{13}\beta}{\beta}\right)+\left(\frac{m_1m_2\beta B\gamma G+m_1m_3aA\gamma G+m_2m_3aA\beta B}{2}-\left(\frac{(r+2)\alpha-\Gamma}{2\alpha}-\frac{Z-m_{12}\gamma}{\gamma}-\frac{\Delta-m_{23}\alpha}{\alpha}+\frac{\Gamma(\gamma-\beta)}{2\beta}\right)\left(\frac{(r+2)\beta-\Gamma}{2\beta}-\frac{Z-m_{12}\beta}{\beta}-\frac{E-m_{13}\alpha}{\alpha}+\frac{\Gamma(\alpha-\gamma)}{2\beta}\right)\right)\frac{1}{\beta\gamma}\sum_{1\le k\le m_{13}}\left(\left(\varepsilon^k-\gamma\right)\left(\frac{\varepsilon^k-\beta}{\varepsilon^k}\right)\right)$$

$$+\frac{m_1m_2\beta B^2\gamma G}{2}\left(\frac{\Gamma-r\beta}{\beta}\right)+\left(\frac{m_1m_2\beta B\gamma G+m_1m_3aA\gamma G+m_2m_3aA\beta B}{2}-\left(\frac{(r+2)\alpha-\Gamma}{2\alpha}-\frac{Z-m_{12}\gamma}{\gamma}-\frac{\Delta-m_{23}\alpha}{\alpha}+\frac{\Gamma(\gamma-\beta)}{2\beta}\right)\left(\frac{(r+2)\beta-\Gamma}{2\beta}-\frac{Z-m_{12}\beta}{\beta}-\frac{E-m_{13}\alpha}{\alpha}+\frac{\Gamma(\alpha-\gamma)}{2\beta}\right)\right)\frac{1}{\beta^2}\sum_{1\le k\le r}\left(\frac{\left(\gamma^k-\beta\right)^2}{\gamma^k}\right)$$

$$+\left(\frac{m_1\gamma G+m_2\beta B+m_3aA}{4}-\left(\frac{(r+2)\alpha-\Gamma}{2\alpha}-\frac{Z-m_{12}\gamma}{\gamma}-\frac{\Delta-m_{23}\alpha}{\alpha}+\frac{\Gamma(\gamma-\beta)}{2\beta}\right)\right)\frac{1}{\beta^3}\sum_{1\le k\le r}\sum_{1\le j\le m_{12}}\left(\frac{\left(\gamma^k-\gamma\right)^2}{\gamma^k}\left(\zeta^j-\beta\right)\right)$$

$$+\left(\frac{m_1\gamma G+m_2\beta B+m_3\alpha A}{4}-\left(\frac{(r+2)\alpha-\Gamma}{2\alpha}-\frac{Z-m_{12}\gamma}{\gamma}-\frac{\Delta-m_{23}\alpha}{\alpha}+\frac{\Gamma(\gamma-\beta)}{2\beta}\right)\right)\frac{1}{\alpha\beta^2}\sum_{1\le k\le r}\sum_{1\le j\le m_{13}}\left(\frac{(\gamma^k-\gamma)^2}{\gamma^k}(\varepsilon^j-\alpha)\right)$$

$$+\left(\frac{m_1\gamma G+m_2\beta B+m_3\alpha A}{4}-\left(\frac{(r+2)\alpha-\Gamma}{2\alpha}-\frac{Z-m_{12}\gamma}{\gamma}-\frac{\Delta-m_{23}\alpha}{\alpha}+\frac{\Gamma(\gamma-\beta)}{2\beta}\right)\right)\frac{1}{\alpha\beta\gamma}\sum_{1\le \kappa<j\le m_{13}}\left((\varepsilon^k-\gamma)(\varepsilon^j-\alpha)\left(\frac{\varepsilon^k-\gamma}{\varepsilon^k}\right)\right)$$

$$+\left(\frac{m_1\gamma G+m_2\beta B+m_3\alpha A}{4}-\left(\frac{(r+2)\alpha-\Gamma}{2\alpha}-\frac{Z-m_{12}\gamma}{\gamma}-\frac{\Delta-m_{23}\alpha}{\alpha}+\frac{\Gamma(\gamma-\beta)}{2\beta}\right)\right)\frac{1}{\beta^3}\sum_{1\le k<j\le r}\left(\frac{(\gamma^k-\gamma)^2}{\gamma^k}(\gamma^j-\beta)\right)$$

$$+\left(\frac{m_1\gamma G+m_2\beta B+m_3\alpha A}{4}-\left(\frac{(r+2)\alpha-\Gamma}{2\alpha}-\frac{Z-m_{12}\gamma}{\gamma}-\frac{\Delta-m_{23}\alpha}{\alpha}+\frac{\Gamma(\gamma-\beta)}{2\beta}\right)\right)\frac{1}{\beta^2\gamma}\sum_{1\le k\le m_{13}}\sum_{1\le j\le m_{12}}\left((\varepsilon^k-\gamma)(\zeta^j-\beta)\ \frac{(\varepsilon^k-\gamma)}{\varepsilon^k}\right)$$

$$+\left(\frac{m_1\gamma G+m_2\beta B+m_3\alpha A}{4}-\left(\frac{(r+2)\alpha-\Gamma}{2\alpha}-\frac{Z-m_{12}\gamma}{\gamma}-\frac{\Delta-m_{23}\alpha}{\alpha}+\frac{\Gamma(\gamma-\beta)}{2\beta}\right)\right)\frac{1}{\alpha\beta\gamma}\sum_{1\le j<k\le m_{13}}\left((\varepsilon^k-\gamma)(\varepsilon^j-\alpha)\left(\frac{\varepsilon^k-\gamma}{\varepsilon^k}\right)\right)$$

$$+\left(\frac{m_1\gamma G+m_2\beta B+m_3\alpha A}{4}-\left(\frac{(r+2)\alpha-\Gamma}{2\alpha}-\frac{Z-m_{12}\gamma}{\gamma}-\frac{\Delta-m_{23}\alpha}{\alpha}+\frac{\Gamma(\gamma-\beta)}{2\beta}\right)\right)\frac{1}{\beta^2\gamma}\sum_{1\le k\le m_{13}}\sum_{1\le j\le r}\left((\varepsilon^k-\gamma)(\gamma^j-\beta)\frac{(\varepsilon^k-\gamma)}{\varepsilon^k}\right)$$

$$+\left(\frac{m_1\gamma G+m_2\beta B+m_3\alpha A}{4}-\left(\frac{(r+2)\alpha-\Gamma}{2\alpha}-\frac{Z-m_{12}\gamma}{\gamma}-\frac{\Delta-m_{23}\alpha}{\alpha}+\frac{\Gamma(\gamma-\beta)}{2\beta}\right)\right)\frac{1}{\beta^3}\sum_{1\le k\le m_{23}}\sum_{1\le j\le r}\left((\delta^k-\beta)(\gamma^j-\beta)\frac{(\varepsilon^k-\gamma)}{\varepsilon^k}\right)$$

$$+\left(\frac{m_1\gamma G+m_2\beta B+m_3\alpha A}{4}-\left(\frac{(r+2)\alpha-\Gamma}{2\alpha}-\frac{Z-m_{12}\gamma}{\gamma}-\frac{\Delta-m_{23}\alpha}{\alpha}+\frac{\Gamma(\gamma-\beta)}{2\beta}\right)\right)\frac{1}{\beta^3}\sum_{1\le k\le m_{23}}\sum_{1\le j\le m_{12}}\left((\zeta^j-\beta)\frac{(\delta^k-\gamma)^2}{\delta^k}\right)$$

$$+\left(\frac{m_1\gamma G+m_2\beta B+m_3\alpha A}{4}-\left(\frac{(r+2)\alpha-\Gamma}{2\alpha}-\frac{Z-m_{12}\gamma}{\gamma}-\frac{\Delta-m_{23}\alpha}{\alpha}+\frac{\Gamma(\gamma-\beta)}{2\beta}\right)\right)\frac{1}{\alpha\beta^2}\sum_{1\le k\le m_{23}}\sum_{1\le j\le m_{13}}\left((\varepsilon^j-\alpha)\frac{(\delta^k-\gamma)^2}{\delta^k}\right)$$

$$+\left(\frac{m_1\gamma G+m_2\beta B+m_3\alpha A}{4}-\left(\frac{(r+2)\alpha-\Gamma}{2\alpha}-\frac{Z-m_{12}\gamma}{\gamma}-\frac{\Delta-m_{23}\alpha}{\alpha}+\frac{\Gamma(\gamma-\beta)}{2\beta}\right)\right)\frac{1}{\beta^3}\sum_{1\le k<j\le r}\left((\gamma^j-\beta)\frac{(\gamma^k-\gamma)^2}{\gamma^k}\right)$$

$+\left(\frac{m_1\gamma G+m_2\beta B+m_3 aA}{4}-\left(\frac{(r+2)\beta-\Gamma}{2\beta}-\frac{Z-m_{12}\beta}{\beta}-\frac{E-m_{13}\alpha}{\alpha}+\frac{\Gamma(\alpha-\gamma)}{2\beta}\right)\right)\frac{1}{\gamma^3}\sum_{1\leq k\leq m_{13}}\sum_{1\leq i\leq m_{12}}\left((\zeta^i-\gamma)\frac{(\varepsilon^k-\gamma)^2}{\varepsilon^k}\right)$

$+\left(\frac{m_1\gamma G+m_2\beta B+m_3 aA}{4}-\left(\frac{(r+2)\beta-\Gamma}{2\beta}-\frac{Z-m_{12}\beta}{\beta}-\frac{E-m_{13}\alpha}{\alpha}+\frac{\Gamma(\alpha-\gamma)}{2\beta}\right)\right)\frac{1}{\alpha\gamma^2}\sum_{1\leq k\leq m_{13}}\sum_{1\leq i\leq r}\left((\gamma^i-\alpha)\left(\frac{(\varepsilon^k-\gamma)^2}{\varepsilon^k}\right)\right)$

$+\left(\frac{m_1\gamma G+m_2\beta B+m_3 aA}{4}-\left(\frac{(r+2)\beta-\Gamma}{2\beta}-\frac{Z-m_{12}\beta}{\beta}-\frac{E-m_{13}\alpha}{\alpha}+\frac{\Gamma(\alpha-\gamma)}{2\beta}\right)\right)\frac{1}{\beta\gamma^2}\sum_{1\leq k\leq m_{23}}\sum_{1\leq i\leq m_{12}}\left((\delta^k-\beta)(\zeta^i-\gamma)\left(\frac{\delta^k-\gamma}{\delta^k}\right)\right)$

$+\left(\frac{m_1\gamma G+m_2\beta B+m_3 aA}{4}-\left(\frac{(r+2)\beta-\Gamma}{2\beta}-\frac{Z-m_{12}\beta}{\beta}-\frac{E-m_{13}\alpha}{\alpha}+\frac{\Gamma(\alpha-\gamma)}{2\beta}\right)\right)\frac{1}{\alpha\beta\gamma}\sum_{1\leq i<k\leq m_{23}}\left((\delta^k-\beta)(\delta^i-\alpha)\left(\frac{\delta^k-\gamma}{\delta^k}\right)\right)$

$+\left(\frac{m_1\gamma G+m_2\beta B+m_3 aA}{4}-\left(\frac{(r+2)\beta-\Gamma}{2\beta}-\frac{Z-m_{12}\beta}{\beta}-\frac{E-m_{13}\alpha}{\alpha}+\frac{\Gamma(\alpha-\gamma)}{2\beta}\right)\right)\frac{1}{\alpha\beta\gamma}\sum_{1\leq k\leq m_{23}}\sum_{1\leq i\leq r}\left((\delta^k-\beta)(\gamma^i-\alpha)\frac{(\delta^k-\gamma)}{\delta^k}\right)$

$+\left(\frac{m_1\gamma G+m_2\beta B+m_3 aA}{4}-\left(\frac{(r+2)\beta-\Gamma}{2\beta}-\frac{Z-m_{12}\beta}{\beta}-\frac{E-m_{13}\alpha}{\alpha}+\frac{\Gamma(\alpha-\gamma)}{2\beta}\right)\right)\frac{1}{\beta\gamma^2}\sum_{1\leq k\leq r}\sum_{1\leq i\leq m_{12}}\left((\gamma^k-\beta)(\zeta^i-\gamma)\left(\frac{\gamma^k-\gamma}{\gamma^k}\right)\right)$

$+\left(\frac{m_1\gamma G+m_2\beta B+m_3 aA}{4}-\left(\frac{(r+2)\beta-\Gamma}{2\beta}-\frac{Z-m_{12}\beta}{\beta}-\frac{E-m_{13}\alpha}{\alpha}+\frac{\Gamma(\alpha-\gamma)}{2\beta}\right)\right)\frac{1}{\alpha\gamma^2}\sum_{1\leq k\leq m_{13}}\sum_{1\leq i\leq m_{23}}\left(\frac{(\varepsilon^k-\gamma)^2}{\varepsilon^k}(\delta^i-\alpha)\right)$

$+\left(\frac{m_1\gamma G+m_2\beta B+m_3 aA}{4}-\left(\frac{(r+2)\beta-\Gamma}{2\beta}-\frac{Z-m_{12}\beta}{\beta}-\frac{E-m_{13}\alpha}{\alpha}+\frac{\Gamma(\alpha-\gamma)}{2\beta}\right)\right)\frac{1}{\alpha\beta\gamma}\sum_{1\leq i<k\leq m_{23}}\left((\delta^k-\beta)(\delta^i-\alpha)\left(\frac{\delta^k-\gamma}{\delta^k}\right)\right)$

$+\left(\frac{m_1\gamma G+m_2\beta B+m_3 aA}{4}-\left(\frac{(r+2)\beta-\Gamma}{2\beta}-\frac{Z-m_{12}\beta}{\beta}-\frac{E-m_{13}\alpha}{\alpha}+\frac{\Gamma(\alpha-\gamma)}{2\beta}\right)\right)\frac{1}{\alpha\beta\gamma}\sum_{1\leq k\leq r}\sum_{1\leq i\leq m_{23}}\left((\gamma^k-\beta)(\delta^i-\alpha)\left(\frac{\gamma^k-\gamma}{\gamma^k}\right)\right)$

$+\left(\frac{m_1\gamma G+m_2\beta B+m_3 aA}{4}-\left(\frac{(r+2)\beta-\Gamma}{2\beta}-\frac{Z-m_{12}\beta}{\beta}-\frac{E-m_{13}\alpha}{\alpha}+\frac{\Gamma(\alpha-\gamma)}{2\beta}\right)\right)\frac{1}{\alpha\beta\gamma}\sum_{1\le i\le m_{12}}\left(\left(\zeta^i-\gamma\right)\left(\zeta^i-\beta\right)\left(\frac{2\alpha-\zeta^i}{\zeta^i}\right)\right)$

$+\left(\frac{m_1\gamma G+m_2\beta B+m_3 aA}{4}-\left(\frac{(r+2)\beta-\Gamma}{2\beta}-\frac{Z-m_{12}\beta}{\beta}-\frac{E-m_{13}\alpha}{\alpha}+\frac{\Gamma(\alpha-\gamma)}{2\beta}\right)\right)\frac{1}{\alpha^2\beta}\sum_{1\le i\le r}\left(\left(\gamma^i-\alpha\right)\left(\gamma^i-\beta\right)\left(\frac{2\alpha-\gamma^i}{\gamma^i}\right)\right).$

Σημειώνουμε ότι οι 25 πιο πάνω όροι αποτελούν απλοποίηση και πρόσθεση μερικών μόνο διαφορών από το άθροισμα $\Delta=\Delta_1+\Delta_2+\cdots+\Delta_{303}$. Παρατηρούμε ότι για την απόδειξη ότι αυτά είναι θετικά αρκεί να αποδείξουμε ότι

$$\frac{m_1m_2\beta B\gamma G+m_1m_3aA\gamma G+m_2m_3aA\beta B}{2}\ge 1 \text{ και, } \frac{m_1\gamma G+m_2\beta B+m_3aA}{4}\ge 1$$

ή,

$$m_1m_2\beta B\gamma G+m_1m_3aA\gamma G+m_2m_3aA\beta B\ge 2 \text{ και, } m_1\gamma G+m_2\beta B+m_3aA\ge 4\,.$$

Για τα πιο πάνω γνωρίζουμε τα ακόλουθα :

- Έχουμε ότι $m_1, m_2, m_3 \ge 1$.

- Επίσης έχουμε ότι $f^\ell=\frac{(m+2)c^\ell-C}{2C}$, όπου $0<f^\ell<1$. Επομένως $0<\frac{(m+2)c^\ell-C}{C}<2$. Επίσης ισχύει $A=\frac{a(m+2)-C}{aC}$ άρα $\alpha A=\frac{a(m+2)-C}{C}$, άρα $0<\alpha A<2$. Με τον ίδιο για τα υπόλοιπα έχουμε ότι $0<aA,\beta B,\gamma G<2$, αφού β$B=\frac{\beta(m+2)-C}{C}$ και $\gamma G=\frac{\gamma(m+2)-C}{C}$.

Συγκεκριμένα, έχουμε χρησιμοποιήσει 50 μόνο όρους από τους υπολοιπόμενους 243 όρους που είχαμε έτσι μας απομένουν ακόμα 193 θετικοί όροι για να τους προσθέσουμε στους 60 όρους τους οποίους θέλουμε να δείξουμε ότι η τελική συνολική πρόσθεση τους θα είναι θετική.

Πιο κάτω βλέπουμε πως απλοποιούμε μερικές από τις υπολοιπόμενες 193 διαφορές για να τις προσθέσουμε με τους 60 όρους.

- Απλοποίηση άλλου όρου για την πρόσθεση του με τον Δ_9

$$\sum_{1\le k\le r} f^k \sum_{1\le i\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+i} \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j}\left(\frac{1}{\varepsilon^i}-\frac{1}{\gamma^k}\right)$$

$$\ge \sum_{1\le k\le r}\sum_{1\le j\le m_{12}} f^k f^{r+m_1+m_2+m_3+j}\left(\frac{1}{\zeta^j}-\frac{1}{\gamma^k}\right)$$

$$= \sum_{1\le k\le r}\sum_{1\le j\le m_{12}} \frac{1}{2}\left(\gamma^k B+\frac{\gamma^k}{\beta}-1\right)\frac{1}{2}\left(B\zeta^j+p_1^j\right)\left(\frac{1}{\zeta^j}-\frac{1}{\gamma^k}\right)(Αφού\ \frac{\gamma^k}{\beta}-1$$
$$\ge \left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right))$$

$$\ge \sum_{1\le k\le r}\sum_{1\le j\le m_{12}} \frac{1}{2}\left(\gamma^k B+p_3^k\right)\frac{1}{2}\left(B\zeta^j+p_1^j\right)\left(\frac{1}{\zeta^j}-\frac{1}{\gamma^k}\right)$$

$$=\frac{1}{4}\sum_{1\le k\le r}\sum_{1\le j\le m_{12}}\left(B^2\gamma^k\zeta^j+B\gamma^k p_1^j+p_3^k B\zeta^j+p_3^k p_1^j\right)\left(\frac{1}{\zeta^j}-\frac{1}{\gamma^k}\right)$$

$$=\frac{1}{4}\sum_{1\le k\le r}\sum_{1\le j\le m_{12}}\left(B^2\gamma^k\zeta^j+B\gamma^k\left(\frac{\zeta^j}{\gamma}-1\right)+\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)B\zeta^j\right.$$
$$\left.+\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\zeta^j}{\gamma}-1\right)\right)\left(\frac{1}{\zeta^j}-\frac{1}{\gamma^k}\right)$$

$$\geq \frac{1}{8}\sum_{1\leq k\leq r}\sum_{1\leq j\leq m_{12}}\left(B^2\gamma^k\zeta^j + B\gamma^k\left(\frac{\zeta^j}{\gamma}-1\right)+\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)B\zeta^j\right.$$
$$\left.+\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\zeta^j}{\gamma}-1\right)\right)\left(\frac{1}{\zeta^j}-\frac{1}{\gamma^k}\right)$$

Την απλοποιημένη αυτή διαφορά θα την προσθέσουμε στη συνέχεια με την απλοποιημένη διαφορά Δ_9 όπως πιο κάτω:

$$\frac{1}{8}\sum_{1\leq k\leq r}\sum_{1\leq j\leq m_{12}}\left(B^2\gamma^k\zeta^j\left(\frac{1}{\zeta^j}-\frac{1}{\gamma^k}\right)+B\gamma^k\left(\frac{\zeta^j}{\gamma}-1\right)\left(\frac{1}{\zeta^j}-\frac{1}{\gamma^k}\right)\right.$$
$$+\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)B\zeta^j\left(\frac{1}{\zeta^j}-\frac{1}{\gamma^k}\right)$$
$$\left.+\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\zeta^j}{\gamma}-1\right)\left(\frac{1}{\zeta^j}-\frac{1}{\gamma^k}\right)\right)$$

$$+\frac{1}{8}\sum_{1\leq k\leq r}\sum_{1\leq j\leq m_{12}}\left(B^3\zeta^j\frac{1}{\gamma^k}S(k)\right.$$
$$+\left(B\left(\frac{\zeta^j}{\gamma}-1\right)\left(\gamma^kB+B\zeta^j+\left(\frac{\zeta^j}{\gamma}-1\right)\right)\frac{1}{\gamma^k}S(k)+\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)B^2\zeta^j\right.$$
$$+\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)B\left(\frac{\zeta^j}{\gamma}-1\right)\frac{1}{\gamma^k}S(k)$$
$$+\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)B\zeta^j\left(\frac{\zeta^j}{\gamma}-1\right)\frac{1}{\gamma^k}S(k)$$
$$\left.\left.-7\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\zeta^j}{\gamma}-1\right)\left(\frac{\zeta^j}{\gamma}-1\right)\frac{1}{\gamma^k}S(k)\right)\right)$$

$$=\frac{1}{8}\sum_{1\leq k\leq r}\sum_{1\leq j\leq m_{12}}\left(B^3\zeta^j\frac{1}{\gamma^k}S(k)+\left(B\left(\frac{\zeta^j}{\gamma}-1\right)\left(\gamma^kB+B\zeta^j+\left(\frac{\zeta^j}{\gamma}-1\right)\right)\frac{1}{\gamma^k}S(k)+\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)B^2\zeta^j+\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)B\left(\frac{\zeta^j}{\gamma}-1\right)\frac{1}{\gamma^k}S(k)+\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)B\zeta^j\left(\frac{\zeta^j}{\gamma}-1\right)\frac{1}{\gamma^k}S(k)+B^2\gamma^k\zeta^j\left(\frac{1}{\zeta^j}-\frac{1}{\gamma^k}\right)+B\gamma^k\left(\frac{\zeta^j}{\gamma}-1\right)\left(\frac{1}{\zeta^j}-\frac{1}{\gamma^k}\right)+\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)B\zeta^j\left(\frac{1}{\zeta^j}-\frac{1}{\gamma^k}\right)+\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\zeta^j}{\gamma}-1\right)\left(\frac{1}{\zeta^j}-\frac{1}{\gamma^k}\right)-7\left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(\frac{\zeta^j}{\gamma}-1\right)\left(\frac{\zeta^j}{\gamma}-1\right)\frac{1}{\gamma^k}S(k)\right)\right)$$

$$= \frac{1}{8}\sum_{1\le k\le r}\sum_{1\le j\le m_{12}}\left(B^3\zeta^j\frac{1}{\gamma^k}S(k) + \left(B\left(\frac{\zeta^j}{\gamma}-1\right)\left(\gamma^k B + B\zeta^j + \left(\frac{\zeta^j}{\gamma}-1\right)\right)\frac{1}{\gamma^k}S(k) + B^2\gamma^k\zeta^j\left(\frac{1}{\zeta^j} - \frac{1}{\gamma^k}\right) + B\gamma^k\left(\frac{\zeta^j}{\gamma}-1\right)\left(\frac{1}{\zeta^j}-\frac{1}{\gamma^k}\right) + \left(\frac{\gamma^k}{2\beta}-\frac{\gamma^k}{2\alpha}+\frac{\gamma^k}{2\gamma}-\frac{1}{2}\right)\left(B^2\zeta^j + B\left(\frac{\zeta^j}{\gamma}-1\right)\frac{1}{\gamma^k}S(k) + B\zeta^j\left(\frac{\zeta^j}{\gamma}-1\right)\frac{1}{\gamma^k}S(k) + B\zeta^j\left(\frac{1}{\zeta^j}-\frac{1}{\gamma^k}\right) + \left(\frac{\zeta^j}{\gamma}-1\right)\left(\frac{1}{\zeta^j}-\frac{1}{\gamma^k}\right) - 7\left(\frac{\zeta^j}{\gamma}-1\right)\left(\frac{\zeta^j}{\gamma}-1\right)\frac{1}{\gamma^k}S(k)\right)\right)\right)$$

Έτσι βλέπουμε πως μπορεί να μεγαλώσει η διαφορά Δ_9 προς το θετικό.

- Απλοποίηση άλλου όρου για την πρόσθεση του με τον Δ_{10}

$$\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k}\sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i}\sum_{1\le j\le r} f^j\left(\frac{1}{\zeta^i}-\frac{1}{\zeta^k}\right)$$

$$\ge \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i}\sum_{1\le j\le r} f^j\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^j}\right)$$

$$\ge \frac{1}{4}\sum_{1\le i\le m_{12}}\sum_{1\le j\le r}\left(G\zeta^i + p_2^i\right)\left(B\gamma^j + p_1^j\right)\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^j}\right)\left(A\varphi o\acute{\upsilon}\ \frac{\gamma^j}{\beta}-1 \ge \left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\right)$$

$$= \frac{1}{4}\sum_{1\le i\le m_{12}}\sum_{1\le j\le r}\left(G\zeta^i B\gamma^j + G\zeta^i p_1^j + p_2^i B\gamma^j + p_2^i p_1^j\right)\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^j}\right)$$

$$= \frac{1}{4}\sum_{1\le i\le m_{12}}\sum_{1\le j\le r}\left(G\zeta^i B\gamma^j + G\zeta^i\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right) + \left(\frac{\zeta^i}{\gamma}-1\right)B\gamma^j + \left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\right)\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^j}\right)$$

$$\ge \frac{1}{8}\sum_{1\le i\le m_{12}}\sum_{1\le j\le r}\left(G\zeta^i B\gamma^j + G\zeta^i\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right) + \left(\frac{\zeta^i}{\gamma}-1\right)B\gamma^j + \left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\right)\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^j}\right)$$

Την απλοποιημένη αυτή διαφορά θα την προσθέσουμε στη συνέχεια με τη διαφορά Δ_{10} όπως πιο κάτω:

$$\frac{1}{8}\sum_{1\le i\le m_{12}}\sum_{1\le j\le r}\left(G\zeta^i B\gamma^j\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^j}\right)+G\zeta^i\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^j}\right)\right.$$
$$+\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^j}\right)B\gamma^j$$
$$\left.+\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^j}\right)\right)$$

$$+\frac{1}{8}\sum_{1\le i\le m_{12}}\sum_{1\le j\le r}\left(\left(B^3\gamma^j\gamma^j\frac{1}{\zeta^i}S(i)+2B^2\gamma^j\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\frac{1}{\zeta^i}S(i)\right.\right.$$
$$+2B\frac{1}{\zeta^i}S(i)\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)$$
$$+B\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\frac{1}{\zeta^i}S(i)$$
$$\left.\left.-\frac{7}{\gamma^j}\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\frac{1}{\zeta^i}S(i)\right)\right)$$

$$=\frac{1}{8}\sum_{1\le i\le m_{12}}\sum_{1\le j\le r}\left(B^3\gamma^j\gamma^j\frac{1}{\zeta^i}S(i)+G\zeta^i B\gamma^j\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^j}\right)+\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^j}\right)B\gamma^j\right.$$
$$+\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(2B^2\gamma^j\frac{1}{\zeta^i}S(i)+2B\frac{1}{\zeta^i}S(i)\left(\frac{\zeta^i}{\gamma}-1\right)\right.$$
$$+B\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\frac{1}{\zeta^i}S(i)+G\zeta^i\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^j}\right)$$
$$\left.\left.+\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^j}\right)-\frac{7}{\gamma^j}\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\frac{1}{\zeta^i}S(i)\right)\right)$$

Έτσι βλέπουμε πως μπορεί να μεγαλώσει η διαφορά Δ_{10} προς το θετικό.

- Απλοποίηση άλλου όρου για την πρόσθεση του με τον Δ_{13}

$$\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le r} f^j\left(\frac{1}{\delta^i}-\frac{1}{\zeta^k}\right)$$

$$\ge \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le r} f^j\left(\frac{1}{\delta^i}-\frac{1}{\zeta^k}\right)$$

$$\ge \frac{1}{4}\sum_{1\le i\le m_{23}}\sum_{1\le j\le r}\left(A\delta^i+p_2^i\right)\left(B\gamma^j+p_1^j\right)\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^j}\right)(\text{Αφού } \frac{\gamma^j}{\beta}-1\ge\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right))$$

$$=\frac{1}{4}\sum_{1\le i\le m_{23}}\sum_{1\le j\le r}\left(A\delta^iB\gamma^j+A\delta^ip_1^j+p_2^iB\gamma^j+p_2^ip_1^j\right)\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^j}\right)$$

$$=\frac{1}{4}\sum_{1\le i\le m_{23}}\sum_{1\le j\le r}\left(AB\delta^i\gamma^j+A\delta^i\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)+\left(\frac{\delta^i}{\alpha}-1\right)B\gamma^j\right.$$
$$\left.+\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\right)\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^j}\right)$$

$$\ge\frac{1}{8}\sum_{1\le i\le m_{23}}\sum_{1\le j\le r}\left(AB\delta^i\gamma^j+A\delta^i\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)+\left(\frac{\delta^i}{\alpha}-1\right)B\gamma^j\right.$$
$$\left.+\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\right)\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^j}\right)$$

Την απλοποιημένη αυτή διαφορά θα την προσθέσουμε στη συνέχεια με την απλοποιημένη διαφορά Δ_{13} πιο κάτω:

$$\frac{1}{8}\sum_{1\le i\le m_{23}}\sum_{1\le j\le r}\left(AB\delta^i\gamma^j\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^j}\right)+A\delta^i\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^j}\right)\right.$$
$$+\left(\frac{\delta^i}{\alpha}-1\right)B\gamma^j\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^j}\right)$$
$$\left.+\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^j}\right)\right)$$

$$+\frac{1}{8}\sum_{1\le i\le m_{23}}\sum_{1\le j\le r}\Bigg(\Bigg(B^2\gamma^j\left(B\gamma^j+\left(\frac{\delta^i}{\alpha}-1\right)\right)\frac{1}{\delta^i}S(i)$$

$$+2B^2\gamma^j\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\frac{1}{\delta^i}S(i)$$

$$+B\left(\frac{\delta^j}{\alpha}-1\right)\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\frac{1}{\delta^i}S(i)$$

$$+B\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\frac{1}{\delta^i}S(i)$$

$$+B\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\frac{1}{\delta^i}S(i)$$

$$-\frac{7}{\gamma^j}\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\frac{1}{\delta^i}S(i)\Bigg)\Bigg)$$

$$=\frac{1}{8}\sum_{1\le i\le m_{23}}\sum_{1\le j\le r}\Bigg(\Bigg(B^2\gamma^j\left(B\gamma^j+\left(\frac{\delta^i}{\alpha}-1\right)\right)\frac{1}{\delta^i}S(i)+2B^2\gamma^j\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\frac{1}{\delta^i}S(i)+B\left(\frac{\delta^j}{\alpha}-1\right)\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\frac{1}{\delta^i}S(i)+B\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\frac{1}{\delta^i}S(i)+B\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\frac{1}{\delta^i}S(i)+AB\delta^i\gamma^j\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^j}\right)+A\delta^i\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^j}\right)+\left(\frac{\delta^i}{\alpha}-1\right)B\gamma^j\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^j}\right)+\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\gamma^j}{2\beta}+\frac{\gamma^j}{2\alpha}-\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^j}\right)-\frac{7}{\gamma^j}\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\frac{1}{\delta^i}S(i)\Bigg)\Bigg)$$

$$=\frac{1}{8}\sum_{1\le i\le m_{23}}\sum_{1\le j\le r}\Bigg(B^2\gamma^j\left(B\gamma^j+\left(\frac{\delta^i}{\alpha}-1\right)\right)\frac{1}{\delta^i}S(i)+AB\delta^i\gamma^j\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^j}\right)+B\gamma^j\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^j}\right)+\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\left(2B^2\gamma^j\frac{1}{\delta^i}S(i)+B\left(\frac{\delta^j}{\alpha}-1\right)\frac{1}{\delta^i}S(i)+B\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\frac{1}{\delta^i}S(i)+B\left(\frac{\delta^i}{\alpha}-1\right)\frac{1}{\delta^i}S(i)+A\delta^i\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^j}\right)+\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{1}{\zeta^i}-\frac{1}{\gamma^j}\right)-\frac{7}{\gamma^j}\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\gamma^j}{2\beta}-\frac{\gamma^j}{2\alpha}+\frac{\gamma^j}{2\gamma}-\frac{1}{2}\right)\frac{1}{\delta^i}S(i)\right)\Bigg)$$

Έτσι βλέπουμε πως μπορεί να μεγαλώσει η διαφορά Δ_{13} προς το θετικό.

- Απλοποίηση άλλου όρου για την πρόσθεση του με τον Δ_{24}

$$\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\delta^i} - \frac{1}{\varepsilon^k}\right)$$

$$\ge \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\delta^i} - \frac{1}{\varepsilon^j}\right)$$

$$= \sum_{1\le i\le m_{23}} \sum_{1\le j\le m_{13}} \frac{1}{2}\left(A\delta^i + p_2^i\right)\frac{1}{2}\left(A\varepsilon^j + p_1^j\right)\left(\frac{1}{\delta^i} - \frac{1}{\varepsilon^j}\right)$$

$$= \frac{1}{4}\sum_{1\le i\le m_{23}} \sum_{1\le j\le m_{13}} \left(A\delta^i A\varepsilon^j + A\delta^i p_1^j + p_2^i A\varepsilon^j + p_2^i p_1^j\right)\left(\frac{1}{\delta^i} - \frac{1}{\varepsilon^j}\right)$$

$$= \frac{1}{4}\sum_{1\le i\le m_{23}} \sum_{1\le j\le m_{13}} \left(A\delta^i A\varepsilon^j + A\delta^i\left(\frac{\varepsilon^j}{\alpha} - 1\right) + \left(\frac{\delta^i}{\alpha} - 1\right)A\varepsilon^j + \left(\frac{\delta^i}{\alpha} - 1\right)\left(\frac{\varepsilon^j}{\alpha} - 1\right)\right)\left(\frac{1}{\delta^i} - \frac{1}{\varepsilon^j}\right)$$

$$\ge \frac{1}{8}\sum_{1\le i\le m_{23}} \sum_{1\le j\le m_{13}} \left(A\delta^i A\varepsilon^j + A\delta^i\left(\frac{\varepsilon^j}{\gamma} - 1\right) + \left(\frac{\delta^i}{\alpha} - 1\right)A\varepsilon^j + \left(\frac{\delta^i}{\alpha} - 1\right)\left(\frac{\varepsilon^j}{\alpha} - 1\right)\right)\left(\frac{1}{\delta^i} - \frac{1}{\varepsilon^j}\right)$$

$$\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\delta^i} - \frac{1}{\zeta^k}\right)$$

$$\ge \sum_{1\le i\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+i} \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\delta^i} - \frac{1}{\varepsilon^j}\right)$$

$$= \sum_{1\le i\le m_{23}} \sum_{1\le j\le m_{13}} \frac{1}{2}\left(A\delta^i + p_2^i\right)\frac{1}{2}\left(A\varepsilon^j + p_1^j\right)\left(\frac{1}{\delta^i} - \frac{1}{\varepsilon^j}\right)$$

$$= \frac{1}{4}\sum_{1\le i\le m_{23}} \sum_{1\le j\le m_{13}} \left(A\delta^i A\varepsilon^j + A\delta^i p_1^j + p_2^i A\varepsilon^j + p_2^i p_1^j\right)\left(\frac{1}{\delta^i} - \frac{1}{\varepsilon^j}\right)$$

$$= \frac{1}{4}\sum_{1\le i\le m_{23}} \sum_{1\le j\le m_{13}} \left(A\delta^i A\varepsilon^j + A\delta^i\left(\frac{\varepsilon^j}{\alpha} - 1\right) + \left(\frac{\delta^i}{\alpha} - 1\right)A\varepsilon^j + \left(\frac{\delta^i}{\alpha} - 1\right)\left(\frac{\varepsilon^j}{\alpha} - 1\right)\right)\left(\frac{1}{\delta^i} - \frac{1}{\varepsilon^j}\right)$$

$$\geq \frac{1}{8}\sum_{1\leq i\leq m_{23}}\sum_{1\leq j\leq m_{13}}\left(A\delta^i A\varepsilon^j + A\delta^i\left(\frac{\varepsilon^j}{\gamma}-1\right)+\left(\frac{\delta^i}{\alpha}-1\right)A\varepsilon^j+\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\varepsilon^j}{\alpha}-1\right)\right)\left(\frac{1}{\delta^i}-\frac{1}{\varepsilon^j}\right)$$

Τις δυο απλοποιημένες αυτές διαφορές θα τις προσθέσουμε στη συνέχεια με τη διαφορά Δ_{24} όπως πιο κάτω :

$$\frac{1}{8}\sum_{1\leq i\leq m_{23}}\sum_{1\leq j\leq m_{13}}\left(2A\delta^i A\varepsilon^j\left(\frac{1}{\delta^i}-\frac{1}{\varepsilon^j}\right)+2A\delta^i\left(\frac{\varepsilon^j}{\alpha}-1\right)\left(\frac{1}{\delta^i}-\frac{1}{\varepsilon^j}\right)\right.$$
$$\left.+2\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{1}{\delta^i}-\frac{1}{\varepsilon^j}\right)A\varepsilon^j+2\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\varepsilon^j}{\alpha}-1\right)\left(\frac{1}{\delta^i}-\frac{1}{\varepsilon^j}\right)\right)$$

$$+\frac{1}{8}\sum_{1\leq i\leq m_{23}}\sum_{1\leq j\leq m_{13}}\left(A^3\varepsilon^j\varepsilon^j\frac{1}{\delta^i}S(i)+2A^2\varepsilon^j\left(\frac{\varepsilon^j}{\gamma}-1\right)\frac{1}{\delta^i}S(i)\right.$$
$$+2A\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\varepsilon^j}{\gamma}-1\right)\left(\frac{\varepsilon^j}{\gamma}-1\right)\frac{1}{\delta^i}S(i)+A\left(\frac{\varepsilon^j}{\gamma}-1\right)\left(\frac{\varepsilon^j}{\gamma}-1\right)\left(\frac{\varepsilon^j}{\gamma}-1\right)\frac{1}{\delta^i}S(i)$$
$$\left.+A^2\varepsilon^j\left(\frac{\varepsilon^j}{\gamma}-1\right)\left(\frac{\delta^i}{\alpha}-1\right)\frac{1}{\delta^i}S(i)-\frac{7}{\varepsilon^j}\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\varepsilon^j}{\alpha}-1\right)\frac{1}{\delta^i}S(i)\right)$$

$$=\frac{1}{8}\sum_{1\leq i\leq m_{23}}\sum_{1\leq j\leq m_{13}}\left(A^3\varepsilon^j\varepsilon^j\frac{1}{\delta^i}S(i)+2A^2\varepsilon^j\left(\frac{\varepsilon^j}{\gamma}-1\right)\frac{1}{\delta^i}S(i)\right.$$
$$+2A\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\varepsilon^j}{\gamma}-1\right)\left(\frac{\varepsilon^j}{\gamma}-1\right)\frac{1}{\delta^i}S(i)$$
$$+A\left(\frac{\varepsilon^j}{\gamma}-1\right)\left(\frac{\varepsilon^j}{\gamma}-1\right)\left(\frac{\varepsilon^j}{\gamma}-1\right)\frac{1}{\delta^i}S(i)+A^2\varepsilon^j\left(\frac{\varepsilon^j}{\gamma}-1\right)\left(\frac{\delta^i}{\alpha}-1\right)\frac{1}{\delta^i}S(i)$$
$$+2A\delta^i A\varepsilon^j\left(\frac{1}{\delta^i}-\frac{1}{\varepsilon^j}\right)+2A\delta^i\left(\frac{\varepsilon^j}{\gamma}-1\right)\left(\frac{1}{\delta^i}-\frac{1}{\varepsilon^j}\right)+2\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{1}{\delta^i}-\frac{1}{\varepsilon^j}\right)A\varepsilon^j$$
$$\left.+2\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\varepsilon^j}{\alpha}-1\right)\left(\frac{1}{\delta^i}-\frac{1}{\varepsilon^j}\right)-\frac{7}{\varepsilon^j}\left(\frac{\delta^i}{\alpha}-1\right)\left(\frac{\varepsilon^j}{\alpha}-1\right)\frac{1}{\delta^i}S(i)\right)$$

Έτσι βλέπουμε πως μπορεί να μεγαλώσει η διαφορά Δ_{24} προς το θετικό.

- Απλοποίηση άλλου όρου για την πρόσθεση του με τον Δ_{37}

$$\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le r} f^i \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j}\left(\frac{1}{\zeta^j}-\frac{1}{\zeta^k}\right)37$$

$$\ge \sum_{1\le i\le r} f^i \sum_{1\le j\le m_{12}} f^{r+m_1+m_2+m_3+j}\left(\frac{1}{\zeta^j}-\frac{1}{\gamma^i}\right)$$

$$= \sum_{1\le i\le r}\sum_{1\le j\le m_{12}} \frac{1}{2}\left(\gamma^i A+\frac{\gamma^i}{a}-1\right)\frac{1}{2}\left(B\zeta^j+p_1^j\right)\left(\frac{1}{\zeta^j}-\frac{1}{\gamma^i}\right)$$

$$\ge \sum_{1\le i\le r}\sum_{1\le j\le m_{12}} \frac{1}{2}\left(\gamma^i A+p_2^i\right)\frac{1}{2}\left(B\zeta^j+p_1^j\right)\left(\frac{1}{\zeta^j}-\frac{1}{\gamma^i}\right)\left(Aφού\ \frac{\gamma^i}{a}-1\ge\left(\frac{\gamma^i}{2a}-\frac{\gamma^i}{2\beta}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\right)$$

$$= \frac{1}{4}\sum_{1\le i\le r}\sum_{1\le j\le m_{12}} \left(\gamma^i AB\zeta^j+\gamma^i Ap_1^j+p_2^i B\zeta^j+p_2^i p_1^j\right)\left(\frac{1}{\zeta^j}-\frac{1}{\gamma^i}\right)$$

$$= \frac{1}{4}\sum_{1\le i\le r}\sum_{1\le j\le m_{12}} \left(AB\gamma^i\zeta^j+\gamma^i A\left(\frac{\zeta^j}{\beta}-1\right)+\left(\frac{\gamma^i}{2a}-\frac{\gamma^i}{2\beta}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)B\zeta^j + \left(\frac{\gamma^i}{2a}-\frac{\gamma^i}{2\beta}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\left(\frac{\zeta^j}{\beta}-1\right)\right)\left(\frac{1}{\zeta^j}-\frac{1}{\gamma^i}\right)$$

$$\ge \frac{1}{8}\sum_{1\le i\le r}\sum_{1\le j\le m_{12}} \left(AB\gamma^i\zeta^j+\gamma^i A\left(\frac{\zeta^j}{\beta}-1\right)+\left(\frac{\gamma^i}{2a}-\frac{\gamma^i}{2\beta}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)B\zeta^j + \left(\frac{\gamma^i}{2a}-\frac{\gamma^i}{2\beta}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\left(\frac{\zeta^j}{\beta}-1\right)\right)\left(\frac{1}{\zeta^j}-\frac{1}{\gamma^i}\right)$$

Την απλοποιημένη αυτή διαφορά θα την προσθέσουμε στη συνέχεια με τη διαφορά Δ_{37} όπως πιο κάτω :

$$\frac{1}{8}\sum_{1\le i\le r}\sum_{1\le j\le m_{12}}\left(AB\gamma^i\zeta^j\left(\frac{1}{\zeta^j}-\frac{1}{\gamma^i}\right)+\gamma^i A\left(\frac{\zeta^j}{\beta}-1\right)\left(\frac{1}{\zeta^j}-\frac{1}{\gamma^i}\right)\right.$$
$$+\left(\frac{\gamma^i}{2a}-\frac{\gamma^i}{2\beta}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)B\zeta^j\left(\frac{1}{\zeta^j}-\frac{1}{\gamma^i}\right)$$
$$\left.+\left(\frac{\gamma^i}{2a}-\frac{\gamma^i}{2\beta}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\left(\frac{\zeta^j}{\beta}-1\right)\left(\frac{1}{\zeta^j}-\frac{1}{\gamma^i}\right)\right)$$

$$+\frac{1}{8}\sum_{1\le i\le r}\sum_{1\le j\le m_{12}}\left(A^3\gamma^i S(j)+A^2\gamma^i\left(\frac{\zeta^j}{\beta}-1\right)\frac{1}{\zeta^j}S(j)+2A^2\zeta^j\left(\frac{\gamma^i}{2\beta}-\frac{\gamma^i}{2\alpha}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\frac{1}{\zeta^j}S(j)\right.$$
$$+2A\left(\frac{\zeta^j}{\beta}-1\right)\left(\frac{\gamma^i}{2\beta}-\frac{\gamma^i}{2\alpha}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\frac{1}{\zeta^j}S(j)$$
$$+A\left(\frac{\gamma^i}{2\beta}-\frac{\gamma^i}{2\alpha}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\left(\frac{\gamma^i}{2\beta}-\frac{\gamma^i}{2\alpha}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\frac{1}{\zeta^j}S(j)$$
$$\left.-\frac{7}{\zeta^j}\left(\frac{\gamma^i}{2\beta}-\frac{\gamma^i}{2\alpha}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\left(\frac{\gamma^i}{2\beta}-\frac{\gamma^i}{2\alpha}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\left(\frac{\zeta^j}{\beta}-1\right)\frac{1}{\zeta^j}S(j)\right)$$

$$\frac{1}{8}\sum_{1\le i\le r}\sum_{1\le j\le m_{12}}\left(A^3\gamma^i S(j)+A^2\gamma^i\left(\frac{\zeta^j}{\beta}-1\right)\frac{1}{\zeta^j}S(j)+AB\gamma^i\zeta^j\left(\frac{1}{\zeta^j}-\frac{1}{\gamma^i}\right)\right.$$
$$+\gamma^i A\left(\frac{\zeta^j}{\beta}-1\right)\left(\frac{1}{\zeta^j}-\frac{1}{\gamma^i}\right)$$
$$+\left(\frac{\gamma^i}{2\beta}-\frac{\gamma^i}{2\alpha}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\left(2A^2\zeta^j\frac{1}{\zeta^j}S(j)+2A\left(\frac{\zeta^j}{\beta}-1\right)\frac{1}{\zeta^j}S(j)\right.$$
$$+A\left(\frac{\gamma^i}{2\beta}-\frac{\gamma^i}{2\alpha}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\frac{1}{\zeta^j}S(j)+B\zeta^j\left(\frac{1}{\zeta^j}-\frac{1}{\gamma^i}\right)+\left(\frac{\zeta^j}{\beta}-1\right)\left(\frac{1}{\zeta^j}-\frac{1}{\gamma^i}\right)$$
$$\left.\left.-\frac{7}{\zeta^j}\left(\frac{\gamma^i}{2\beta}-\frac{\gamma^i}{2\alpha}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\left(\frac{\zeta^j}{\beta}-1\right)\frac{1}{\zeta^j}S(j)\right)\right)$$

Έτσι βλέπουμε πως μπορεί να μεγαλώσει η διαφορά Δ_{37} προς το θετικό.

- Απλοποίηση άλλου όρου για την πρόσθεση του με τον Δ_{38}

$$\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i} \sum_{1\le j\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+j}\left(\frac{1}{\delta^j}-\frac{1}{\varepsilon^k}\right)$$

$$\ge \sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le m_{12}} f^{r+m_1+m_2+m_3+i}\left(\frac{1}{\varepsilon^k}-\frac{1}{\zeta^i}\right)$$

$$=\frac{1}{4}\sum_{1\le k\le m_{13}}\sum_{1\le i\le m_{12}}\left(G\varepsilon^k+p_3^k\right)\left(G\zeta^i+p_2^i\right)\left(\frac{1}{\varepsilon^k}-\frac{1}{\zeta^i}\right)$$

$$=\frac{1}{4}\sum_{1\le k\le m_{13}}\sum_{1\le i\le m_{12}}\left(G\varepsilon^k G\zeta^i+G\varepsilon^k p_2^i+p_3^k G\zeta^i+p_3^k p_2^i\right)\left(\frac{1}{\varepsilon^k}-\frac{1}{\zeta^i}\right)$$

$$=\frac{1}{4}\sum_{1\le k\le m_{13}}\sum_{1\le i\le m_{12}}\left(G\varepsilon^k G\zeta^i+G\varepsilon^k\left(\frac{\zeta^i}{\gamma}-1\right)+\left(\frac{\varepsilon^k}{\gamma}-1\right)G\zeta^i+\left(\frac{\varepsilon^k}{\gamma}-1\right)\left(\frac{\zeta^i}{\gamma}-1\right)\right)\left(\frac{1}{\varepsilon^k}-\frac{1}{\zeta^i}\right)$$

$$\ge\frac{1}{8}\sum_{1\le k\le m_{13}}\sum_{1\le i\le m_{12}}\left(G\varepsilon^k G\zeta^i+G\varepsilon^k\left(\frac{\zeta^i}{\gamma}-1\right)+\left(\frac{\varepsilon^k}{\gamma}-1\right)G\zeta^i+\left(\frac{\varepsilon^k}{\gamma}-1\right)\left(\frac{\zeta^i}{\gamma}-1\right)\right)\left(\frac{1}{\varepsilon^k}-\frac{1}{\zeta^i}\right)$$

Την απλοποιημένη αυτή διαφορά θα την προσθέσουμε στη συνέχεια με τη διαφορά Δ_{38} όπως πιο κάτω :

$$\frac{1}{8}\sum_{1\le k\le m_{13}}\sum_{1\le i\le m_{12}}\left(G\varepsilon^k G\zeta^i\left(\frac{1}{\varepsilon^k}-\frac{1}{\zeta^i}\right)+G\varepsilon^k\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{1}{\varepsilon^k}-\frac{1}{\zeta^i}\right)+\left(\frac{\varepsilon^k}{\gamma}-1\right)G\zeta^i\left(\frac{1}{\varepsilon^k}-\frac{1}{\zeta^i}\right)\right.$$
$$\left.+\left(\frac{\varepsilon^k}{\gamma}-1\right)\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{1}{\varepsilon^k}-\frac{1}{\zeta^i}\right)\right)$$

$$+\frac{1}{8}\sum_{1\le i\le m_{12}}\sum_{1\le k\le m_{13}}\left(A^3\zeta^i S(k)+A^2\left(\frac{\zeta^i}{\gamma}-1\right)S(k)+2A^2\zeta^i\left(\frac{\varepsilon^k}{\gamma}-1\right)\frac{1}{\varepsilon^k}S(k)\right.$$
$$+2A\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{\varepsilon^k}{\gamma}-1\right)\frac{1}{\varepsilon^k}S(k)+A\zeta^i\left(\frac{\varepsilon^k}{\gamma}-1\right)\left(\frac{\varepsilon^k}{\gamma}-1\right)\frac{1}{\varepsilon^k}S(k)$$
$$\left.-\frac{7}{\varepsilon^k}\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{\varepsilon^k}{\alpha}-1\right)\left(\frac{\varepsilon^k}{\gamma}-1\right)\frac{1}{\varepsilon^k}S(k)\right)$$

$$=\frac{1}{8}\sum_{1\le i\le m_{12}}\sum_{1\le k\le m_{13}}\left(A^3\zeta^i S(k)+A^2\left(\frac{\zeta^i}{\gamma}-1\right)S(k)+G\varepsilon^k G\zeta^i\left(\frac{1}{\varepsilon^k}-\frac{1}{\zeta^i}\right)\right.$$
$$+G\varepsilon^k\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{1}{\varepsilon^k}-\frac{1}{\zeta^i}\right)$$
$$+\left(\frac{\varepsilon^k}{\gamma}-1\right)\left(2A^2\zeta^i\frac{1}{\varepsilon^k}S(k)+2A\left(\frac{\zeta^i}{\gamma}-1\right)\frac{1}{\varepsilon^k}S(k)\right.$$
$$+A\zeta^i\left(\frac{\varepsilon^k}{\gamma}-1\right)\frac{1}{\varepsilon^k}S(k)+G\zeta^i\left(\frac{1}{\varepsilon^k}-\frac{1}{\zeta^i}\right)+\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{1}{\varepsilon^k}-\frac{1}{\zeta^i}\right)$$
$$\left.\left.-\frac{7}{\varepsilon^k}\left(\frac{\zeta^i}{\gamma}-1\right)\left(\frac{\varepsilon^k}{\alpha}-1\right)\frac{1}{\varepsilon^k}S(k)\right)\right)$$

Έτσι βλέπουμε πως μπορεί να μεγαλώσει η διαφορά Δ_{38} προς το θετικό.

- Απλοποίηση άλλου όρου για την πρόσθεση του με τον Δ_{39}

$$\sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le r} f^i \sum_{1\le j\le m_{23}} f^{r+m_1+m_2+m_3+m_{12}+m_{13}+j}\left(\frac{1}{\delta^j}-\frac{1}{\varepsilon^k}\right)39$$

$$\ge \sum_{1\le k\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+k} \sum_{1\le i\le r} f^i\left(\frac{1}{\varepsilon^k}-\frac{1}{\gamma^i}\right)$$

$$= \sum_{1\le k\le m_{13}} \sum_{1\le i\le r} \frac{1}{2}\left(G\varepsilon^k+p_3^k\right)\frac{1}{2}\left(\gamma^i A+\frac{\gamma^i}{a}-1\right)\left(\frac{1}{\varepsilon^k}-\frac{1}{\gamma^i}\right)$$

$$\ge \sum_{1\le k\le m_{13}} \sum_{1\le i\le r} \frac{1}{2}\left(G\varepsilon^k+p_3^k\right)\frac{1}{2}\left(\gamma^i A+p_2^i\right)\left(\frac{1}{\varepsilon^k}-\frac{1}{\gamma^i}\right)\left(A\varphi o\acute{\upsilon}\ \frac{\gamma^i}{a}-1\ge\left(\frac{\gamma^i}{2a}-\frac{\gamma^i}{2\beta}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\right)$$

$$=\frac{1}{4}\sum_{1\le k\le m_{13}} \sum_{1\le i\le r}\left(G\varepsilon^k\gamma^i A+G\varepsilon^k p_2^i+p_3^k\gamma^i A+p_3^k p_2^i\right)\left(\frac{1}{\varepsilon^k}-\frac{1}{\gamma^i}\right)$$

$$=\frac{1}{4}\sum_{1\le k\le m_{13}} \sum_{1\le i\le r}\left(GA\varepsilon^k\gamma^i+G\varepsilon^k\left(\frac{\gamma^i}{2a}-\frac{\gamma^i}{2\beta}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)+\left(\frac{\varepsilon^j}{\gamma}-1\right)\gamma^i A\right.$$
$$\left.+\left(\frac{\varepsilon^j}{\gamma}-1\right)\left(\frac{\gamma^i}{2a}-\frac{\gamma^i}{2\beta}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\right)\left(\frac{1}{\varepsilon^k}-\frac{1}{\gamma^i}\right)$$

$$\ge\frac{1}{8}\sum_{1\le k\le m_{13}} \sum_{1\le i\le r}\left(GA\varepsilon^k\gamma^i+G\varepsilon^k\left(\frac{\gamma^i}{2a}-\frac{\gamma^i}{2\beta}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)+\left(\frac{\varepsilon^j}{\gamma}-1\right)\gamma^i A\right.$$
$$\left.+\left(\frac{\varepsilon^j}{\gamma}-1\right)\left(\frac{\gamma^i}{2a}-\frac{\gamma^i}{2\beta}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\right)\left(\frac{1}{\varepsilon^k}-\frac{1}{\gamma^i}\right)$$

Την απλοποιημένη αυτή διαφορά θα την προσθέσουμε στη συνέχεια με τη διαφορά Δ_{39} όπως πιο κάτω :

$$\frac{1}{8}\sum_{1\le k\le m_{13}}\sum_{1\le i\le r}\left(GA\varepsilon^k\gamma^i\left(\frac{1}{\varepsilon^k}-\frac{1}{\gamma^i}\right)+G\varepsilon^k\left(\frac{\gamma^i}{2a}-\frac{\gamma^i}{2\beta}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\left(\frac{1}{\varepsilon^k}-\frac{1}{\gamma^i}\right)\right.$$
$$+\left(\frac{1}{\varepsilon^k}-\frac{1}{\gamma^i}\right)\left(\frac{\varepsilon^j}{\gamma}-1\right)\gamma^iA$$
$$\left.+\left(\frac{\varepsilon^j}{\gamma}-1\right)\left(\frac{\gamma^i}{2a}-\frac{\gamma^i}{2\beta}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\left(\frac{1}{\varepsilon^k}-\frac{1}{\gamma^i}\right)\right)$$
$$+\frac{1}{8}\sum_{1\le i\le r}\sum_{1\le k\le m_{13}}\left(A^3\gamma^iS(k)+A^2\left(\frac{\gamma^i}{2a}-\frac{\gamma^i}{2\beta}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)S(k)\right.$$
$$+2A^2\gamma^i\left(\frac{\varepsilon^k}{\gamma}-1\right)\frac{1}{\varepsilon^k}S(k)$$
$$+2A\left(\frac{\gamma^i}{2a}-\frac{\gamma^i}{2\beta}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\left(\frac{\varepsilon^k}{\gamma}-1\right)\frac{1}{\varepsilon^k}S(k)$$
$$+A\gamma^i\left(\frac{\varepsilon^k}{\gamma}-1\right)\left(\frac{\varepsilon^k}{\gamma}-1\right)\frac{1}{\varepsilon^k}S(k)$$
$$\left.-\frac{7}{\varepsilon^k}\left(\frac{\gamma^i}{2a}-\frac{\gamma^i}{2\beta}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\left(\frac{\varepsilon^k}{\alpha}-1\right)\left(\frac{\varepsilon^k}{\gamma}-1\right)\frac{1}{\varepsilon^k}S(k)\right)$$

$$=\frac{1}{8}\sum_{1\le i\le r}\sum_{1\le k\le m_{13}}\left(A^3\gamma^iS(k)+GA\varepsilon^k\gamma^i\left(\frac{1}{\varepsilon^k}-\frac{1}{\gamma^i}\right)+\left(\frac{1}{\varepsilon^k}-\frac{1}{\gamma^i}\right)\left(\frac{\varepsilon^j}{\gamma}-1\right)\gamma^iA\right.$$
$$+2A^2\gamma^i\left(\frac{\varepsilon^k}{\gamma}-1\right)\frac{1}{\varepsilon^k}S(k)$$
$$+\left(\frac{\gamma^i}{2a}-\frac{\gamma^i}{2\beta}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\left(A^2S(k)+2A\left(\frac{\varepsilon^k}{\gamma}-1\right)\frac{1}{\varepsilon^k}S(k)\right.$$
$$+A\gamma^i\left(\frac{\varepsilon^k}{\gamma}-1\right)\left(\frac{\varepsilon^k}{\gamma}-1\right)\frac{1}{\varepsilon^k}S(k)+G\varepsilon^k\left(\frac{1}{\varepsilon^k}-\frac{1}{\gamma^i}\right)$$
$$\left.\left.+\left(\frac{\varepsilon^j}{\gamma}-1\right)\left(\frac{1}{\varepsilon^k}-\frac{1}{\gamma^i}\right)-\frac{7}{\varepsilon^k}\left(\frac{\varepsilon^k}{\alpha}-1\right)\left(\frac{\varepsilon^k}{\gamma}-1\right)\frac{1}{\varepsilon^k}S(k)\right)\right)$$

Έτσι βλέπουμε πως μπορεί να μεγαλώσει η διαφορά Δ_{39} προς το θετικό.

- Απλοποίηση άλλου όρου για την πρόσθεση του με τον Δ_{40}

$$\sum_{1\le k\le m_{12}} f^{r+m_1+m_2+m_3+k} \sum_{1\le i\le r} f^i \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\varepsilon^j}-\frac{1}{\zeta^k}\right)40$$

$$\ge \sum_{1\le i\le r} f^i \sum_{1\le j\le m_{13}} f^{r+m_1+m_2+m_3+m_{12}+j}\left(\frac{1}{\varepsilon^j}-\frac{1}{\gamma^i}\right)$$

$$=\sum_{1\le i\le r}\sum_{1\le j\le m_{13}} \frac{1}{2}\left(\gamma^i A+\frac{\gamma^i}{a}-1\right)\frac{1}{2}\left(A\varepsilon^j+p_1^j\right)\left(\frac{1}{\varepsilon^j}-\frac{1}{\zeta^k}\right)$$

$$\ge \frac{1}{4}\sum_{1\le i\le r}\sum_{1\le j\le m_{13}} \left(\gamma^i A+p_2^i\right)\left(A\varepsilon^j+p_1^j\right)\left(\frac{1}{\varepsilon^j}-\frac{1}{\gamma^i}\right)\left(A\varphi o\acute{\upsilon}\ \frac{\gamma^i}{a}-1\ge\left(\frac{\gamma^i}{2a}-\frac{\gamma^i}{2\beta}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\right)$$

$$=\frac{1}{4}\sum_{1\le i\le r}\sum_{1\le j\le m_{13}} \left(A^2\gamma^i\varepsilon^j+\gamma^i A\left(\frac{\varepsilon^j}{\alpha}-1\right)+\left(\frac{\gamma^i}{2a}-\frac{\gamma^i}{2\beta}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)A\varepsilon^j + \left(\frac{\gamma^i}{2a}-\frac{\gamma^i}{2\beta}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\left(\frac{\varepsilon^j}{\alpha}-1\right)\right)\left(\frac{1}{\varepsilon^j}-\frac{1}{\gamma^i}\right)$$

$$\ge \frac{1}{8}\sum_{1\le i\le r}\sum_{1\le j\le m_{13}} \left(A^2\gamma^i\varepsilon^j\left(\frac{1}{\varepsilon^j}-\frac{1}{\gamma^i}\right)+\gamma^i A\left(\frac{\varepsilon^j}{\alpha}-1\right)\left(\frac{1}{\varepsilon^j}-\frac{1}{\gamma^i}\right)+\left(\frac{\gamma^i}{2a}-\frac{\gamma^i}{2\beta}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)A\varepsilon^j\left(\frac{1}{\varepsilon^j}-\frac{1}{\gamma^i}\right) + \left(\frac{\gamma^i}{2a}-\frac{\gamma^i}{2\beta}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\left(\frac{\varepsilon^j}{\alpha}-1\right)\left(\frac{1}{\varepsilon^j}-\frac{1}{\gamma^i}\right)\right)$$

Την απλοποιημένη αυτή διαφορά θα την προσθέσουμε στη συνέχεια με τη διαφορά Δ_{40} όπως πιο κάτω :

$$\frac{1}{8}\sum_{1\le i\le r}\sum_{1\le j\le m_{13}} \left(A^2\gamma^i\varepsilon^j\left(\frac{1}{\varepsilon^j}-\frac{1}{\gamma^i}\right)+\gamma^i A\left(\frac{\varepsilon^j}{\alpha}-1\right)\left(\frac{1}{\varepsilon^j}-\frac{1}{\gamma^i}\right)+\left(\frac{\gamma^i}{2a}-\frac{\gamma^i}{2\beta}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)A\varepsilon^j\left(\frac{1}{\varepsilon^j}-\frac{1}{\gamma^i}\right) + \left(\frac{\gamma^i}{2a}-\frac{\gamma^i}{2\beta}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\left(\frac{\varepsilon^j}{\alpha}-1\right)\left(\frac{1}{\varepsilon^j}-\frac{1}{\gamma^i}\right)\right)$$

$$+\frac{1}{8}\sum_{1\le i\le r}\sum_{1\le j\le m_{13}}\left(A^3\gamma^i S(j)+A^2\gamma^i\left(\frac{\varepsilon^j}{\alpha}-1\right)\frac{1}{\varepsilon^j}S(j)+\left(\frac{\gamma^i}{2\beta}-\frac{\gamma^i}{2\alpha}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)2A^2S(j)\right.$$
$$+2A\left(\frac{\gamma^i}{2\beta}-\frac{\gamma^i}{2\alpha}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\left(\frac{\varepsilon^j}{\alpha}-1\right)\frac{1}{\varepsilon^j}S(j)$$
$$+A\left(\frac{\gamma^i}{2\beta}-\frac{\gamma^i}{2\alpha}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\left(\frac{\gamma^i}{2\beta}-\frac{\gamma^i}{2\alpha}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\frac{1}{\varepsilon^j}S(j)$$
$$\left.-\frac{7}{\varepsilon^j}\left(\frac{\gamma^i}{2\beta}-\frac{\gamma^i}{2\alpha}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\left(\frac{\gamma^i}{2a}-\frac{\gamma^i}{2\beta}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\left(\frac{\varepsilon^j}{\alpha}-1\right)\frac{1}{\varepsilon^j}S(j)\right)$$

$$=\frac{1}{8}\sum_{1\le i\le r}\sum_{1\le j\le m_{13}}\left(A^3\gamma^i S(j)+A^2\gamma^i\left(\frac{\varepsilon^j}{\alpha}-1\right)\frac{1}{\varepsilon^j}S(j)+\gamma^i A\left(\frac{\varepsilon^j}{\alpha}-1\right)\left(\frac{1}{\varepsilon^j}-\frac{1}{\gamma^i}\right)\right.$$
$$+\left(\frac{\gamma^i}{2a}-\frac{\gamma^i}{2\beta}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)A\varepsilon^j\left(\frac{1}{\varepsilon^j}-\frac{1}{\gamma^i}\right)+\left(\frac{\gamma^i}{2a}-\frac{\gamma^i}{2\beta}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\left(\frac{\varepsilon^j}{\alpha}-1\right)\left(\frac{1}{\varepsilon^j}-\frac{1}{\gamma^i}\right)$$
$$+\left(\frac{\gamma^i}{2\beta}-\frac{\gamma^i}{2\alpha}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\left(2A^2S(j)+2A\left(\frac{\varepsilon^j}{\alpha}-1\right)\frac{1}{\varepsilon^j}S(j)+A\left(\frac{\gamma^i}{2\beta}-\frac{\gamma^i}{2\alpha}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\frac{1}{\varepsilon^j}S(j)\right.$$
$$\left.\left.+A^2\gamma^i\varepsilon^j\left(\frac{1}{\varepsilon^j}-\frac{1}{\gamma^i}\right)-\frac{7}{\varepsilon^j}\left(\frac{\gamma^i}{2a}-\frac{\gamma^i}{2\beta}+\frac{\gamma^i}{2\gamma}-\frac{1}{2}\right)\left(\frac{\varepsilon^j}{\alpha}-1\right)\frac{1}{\varepsilon^j}S(j)\right)\right)$$

Έτσι βλέπουμε πως μπορεί να μεγαλώσει η διαφορά Δ_{40} προς το θετικό.

Κεφάλαιο 9

Συμπεράσματα

9.1 Σύνοψη

Η εργασία αυτή ξεκίνησε με μια αναφορά στην Ισορροπία Nash και το παιχνίδι δρομολόγησης στο οποίο έχουμε αυτή την Ισορροπία. Στην συνέχεια παρουσιάσαμε κάποιους ορισμούς και σημαντικές έννοιες που έχουν να κάνουν με το πλαίσιο εργασίας της περίπτωσης που μελετούμε, τις οποίες και θα χρησιμοποιούσαμε μετά. Σημείο αναφοράς της εργασίας αυτής αποτελεί η Εικασία της Πλήρως Μικτής Ισορροπίας Nash που είδαμε στο 1.2, την οποία και προσπαθήσαμε να αποδείξουμε.

Η ορθότητα της εικασίας αυτής αποδείκτηκε στη βασική βιβλιογραφία **[1]** όπου είχαμε την περίπτωση των 2 παικτών στο συγκεκριμένο παιχνίδι δρομολόγησης σε αυθαίρετο αριθμό συνδέσεων. Αυτή η εργασία αποτελεί επέκταση της προηγούμενης, όπου προσπαθήσαμε να αποδείξουμε το ίδιο, με 3 παίκτες αυτή τη φορά.

Συγκεκριμένα, με απλά λόγια, η εικασία λέει ότι η επιλογή ολων των συνδέσεων με θετική πιθανότητα από όλους τους παίκτες μεγιστοποιεί το Κοινωνικό Κόστος όπως αυτό ορίζεται στη βασική βιβλιογραφία **[1]**, με αποτέλεσμα η περίπτωση της Πλήρως Μικτής Ισορροπίας Nash να αποτελεί το χείριστο σενάριο. Για να αποδείξουμε την εικασία, προσπαθήσαμε να δείξουμε ότι η διαφορά μεταξύ του Κοινωνικού Κόστους που έχουμε στην περίπτωση της ΠΜΙΝ και αυτού μιας Αυθαίρετης Ισορροπίας Nash, διάφορη της ΠΜΙΝ, είναι θετική.

Αυτό που κάναμε αρχικά, είναι να δείξουμε την βασική δομή του μοντέλου που μελετούμε. Στη συνέχεια αποδείξαμε κάποιες μαθηματικές προτάσεις οι οποίες θα μας βοηθούσαν στην έκφραση του Κοινωνικού Κόστους της οποιασδήποτε Μικτής Ανάθεσης. Έτσι μπορέσαμε και εκφράσαμε το Κοινωνικό Κόστος της Πλήρως

Μικτής Ισορροπίας Nash και αυτό της Αυθαίρετης Ισορροπίας Nash. Προχωρήσαμε εκτελώντας τη διαφορά μεταξύ τους όπου καταφέραμε να εξάξουμε κάποια πολύ ενδιαφέρον συμπεράσματα.

9.1 Συμπεράσματα

Ήδη από την έκφραση του Κοινωνικού Κόστους που κάναμε για τις δυο περιπτώσεις της ΠΜΙΝ και Αυθαίρετης Ισορροπίας Nash, αρχίσαμε να βλέπουμε τη διαισθητική πλευρά της εικασίας. Είδαμε να δημιουργούνται συγκεκριμένα 302 όροι οι οποίοι εκφράζουν το Κοινωνικό Κόστος της ΠΜΙΝ, και 59 όροι οι οποίοι εκφράζουν το Κοινωνικό Κόστος της δεύτερης περίπτωσης. Αυτό είναι πολύ ενδιαφέρον αφού ήδη το θετικό της διαφοράς αυτής των δυο περιπτώσεων έχει αρχίσει να φαίνεται από νωρίς.

Επίσης είδαμε τη δυνατότητα που έχουμε να συνδιάσουμε με άνετο τρόπο τους όρους αυτούς, έτσι ώστε να προχωρήσουμε στην απλοποίηση και πρόσθεσή τους, βλέποντας τους πίνακες που φτιάξαμε(βλέπε Παράρτημα Α, Β) και παίρνοντας τις περιπτώσεις που είναι ίδιες μεταξύ τους.

Μόνο εμπόδιο σ’ αυτό, στάθηκε ο χρόνος και η απροσδόκητη πολυπλοκότητα που προέκυψε. Είναι προφανές πως χρειάζεται αρκετός χρόνος για την διαχείρηση τόσων πολλών όρων και την σωστή ολοκλήρωση της απόδειξης, γεγονός στο οποίο οφείλεται και η μη ολοκληρωμένη απόδειξη που είδαμε στην εργασία αυτή.

Όμως κάθε άλλο παρά αισιόδοξους μας αφήνει το γεγονός, πως με προσεκτική συνέχεια της εργασίας αυτής, κάποιος μπορεί να δει, ότι εύκολα μπορεί να φτάσει στην ολοκλήρωση της απόδειξης αυτής καθώς επίσης η συνέχεια αυτή αφήνει τις υποψίες για μια μελλοντική και πολλά υποσχόμενη δουλειά.

9.3 Μελλοντική Δουλειά

Με την ολοκλήρωση της απόδειξης, κάποιος θα μπορούσε να συνεχίσει στην επέκταση του μοντέλου με διάφορους τρόπους, αλλάζοντας την έννοια των συνδέσεων από συσχετιζόμενες σε ταυτόσημες, όπου όλες οι συνδέσεις θα έχουν πλέον ίδια χωρητικότητα. Επίσης θα μπορούσαν να μελετηθούν οι περιπτώσεις με περισσότερους από τρεις παίκτες.

Ακόμα, ενδιαφέρον παρουσιάζει η περίπτωση όπου έχουμε αυθαίρετο αριθμό ταυτόσημων παικτών και τρεις συνδέσεις.

Βιβλιογραφία

[1] Thomas Lucking, Marios Mavronicolas, Burkhand Monien, Manuel Rode, Paul Spirakis and Imrich Vrto "Which is the Worst-Case Nash Equilibrium?," ALCOM-FT, July 29, 2009

[2] M. Mavronicolas and P. Spirakis, "The Price of Selfish Routing," Algorithmica, Vol. 48, No. 1, pp. 91-126, 2007

Παράρτημα Α

Ανάλυση του όρου $\sum_{1\le k<j<i\le m} p_3^k\, p_2^i\, p_1^j \left(\frac{1}{c^i}-\frac{1}{c^k}\right)$ στην εφαρμογή του λήμματος 4.1 στην περίπτωση της αυθαίρετης Ισορροπίας Nash :

Ο παίκτης 1 επιλέγει την σύνδεση j,ο παίκτης 2 την σύνδεση i και ο παίκτης 3 την σύνδεση k. Οι παίκτες επιλέγουν συνδέσεις σύμφωνα με τον περιορισμό $i > j > k$, έτσι σύμφωνα με την αρχική υπόθεση $c^1 \ge c^2 \ge \cdots \ge c^m$ θα έχουμε ότι $c^i < c^j < c^k$. Επίσης έχουμε τον περιορισμό ότι ο καθένας μπορεί να επιλέξει συνδέσεις για τις οποίες η πιθανότητα για επιλογή τους από τον αντίστοιχο παίκτη είναι θετική(δηλαδή οι αντίστοιχες συνδέσεις ανήκουν στο support του):

Πίνακας Α-1

Παίκτης 2(σύνδεση i)	**Παίκτης 1(σύνδεση j)**	**Παίκτης 3(σύνδεση k)**
β	γ	$\delta^k, \varepsilon^k, \gamma^k$
	ζ^j	γ^k
	ε^j	ε^k, γ^k
	γ^j	γ^k
ζ^i	ζ^j	γ^k
	γ^j	γ^k
δ^i	ζ^j	γ^k
	ε^j	γ^k
	γ^j	γ^k
γ^i	γ^j	γ^k

Παρατηρούμε ότι προκύπτουν 13 περιπτώσεις οι οποίες και αποτελούν τον πρώτο όρο.

Ανάλυση του όρου $\sum_{1\le j<k<i\le m} p_3^k\, p_2^i\, p_1^j \left(\frac{1}{c^i}-\frac{1}{c^k}\right)$ στην εφαρμογή του λήμματος 4.1 στην περίπτωση της αυθαίρετης Ισορροπίας Nash :

Ο παίκτης 1 επιλέγει την σύνδεση j,ο παίκτης 2 την σύνδεση i και ο παίκτης 3 την σύνδεση k. Οι παίκτες επιλέγουν συνδέσεις σύμφωνα με τον περιορισμό $i > k > j$, έτσι σύμφωνα με την αρχική υπόθεση $c^1 \ge c^2 \ge \cdots \ge c^m$ θα έχουμε ότι $c^i < c^k < c^j$. Επίσης έχουμε τον περιορισμό ότι ο καθένας μπορεί να επιλέξει συνδέσεις για τις οποίες η πιθανότητα για επιλογή τους από τον αντίστοιχο παίκτη είναι θετική(δηλαδή οι αντίστοιχες συνδέσεις ανήκουν στο support του):

Πίνακας Α-2

Παίκτης 2(σύνδεση i)	**Παίκτης 3(σύνδεση k)**	**Παίκτης 1(σύνδεση j)**
β	ε^k	$\zeta^j, \varepsilon^j, \gamma^j$
	δ^k	$\zeta^j, \varepsilon^j, \gamma^j$
	γ^k	γ^j
ζ^i	γ^k	γ^j
δ^i	ε^k	$\zeta^j, \varepsilon^j, \gamma^j$
	δ^k	$\zeta^j, \varepsilon^j, \gamma^j$
	γ^k	γ^j
γ^i	γ^j	γ^j

Παρατηρούμε ότι προκύπτουν 16 περιπτώσεις οι οποίες αποτελούν και το δεύτερο όρο.

Ανάλυση του όρου $\sum_{1\le k<i<j\le m} p_3^k \, p_2^i \, p_1^j \left(\frac{1}{c^j} - \frac{1}{c^k}\right)$ στην εφαρμογή του λήμματος 4.1 στην περίπτωση της αυθαίρετης Ισορροπίας Nash :

Ο παίκτης 1 επιλέγει την σύνδεση j,ο παίκτης 2 την σύνδεση i και ο παίκτης 3 την σύνδεση k. Οι παίκτες επιλέγουν συνδέσεις σύμφωνα με τον περιορισμό $j > i > k$, έτσι σύμφωνα με την αρχική υπόθεση $c^1 \ge c^2 \ge \cdots \ge c^m$ θα έχουμε ότι $c^j < c^k < c^i$. Επίσης έχουμε τον περιορισμό ότι ο καθένας μπορεί να επιλέξει συνδέσεις για τις οποίες η πιθανότητα για επιλογή τους από τον αντίστοιχο παίκτη είναι θετική(δηλαδή οι αντίστοιχες συνδέσεις ανήκουν στο support του):

Πίνακας Α-3

Παίκτης 1(σύνδεση j)	**Παίκτης 2(σύνδεση i)**	**Παίκτης 3(σύνδεση k)**
γ	ζ^i	γ^k
	δ^i	$\varepsilon^k, \delta^k, \gamma^k$
	γ^i	γ^k
ζ^j	ζ^i	γ^k
	γ^i	γ^k
ε^j	ζ^i	γ^k

	γ^i	γ^k
γ^j	γ^i	γ^k

Παρατηρούμε ότι προκύπτουν 10 περιπτώσεις οι οποίες αποτελούν και τον τρίτο όρο.

Ανάλυση του όρου $\sum_{1\le i<k<j\le m} p_3^k\, p_2^i\, p_1^j \left(\frac{1}{c^j} - \frac{1}{c^k}\right)$ στην εφαρμογή του λήμματος 4.1 στην περίπτωση της αυθαίρετης Ισορροπίας Nash :

Ο παίκτης 1 επιλέγει την σύνδεση j,ο παίκτης 2 την σύνδεση i και ο παίκτης 3 την σύνδεση k. Οι παίκτες επιλέγουν συνδέσεις σύμφωνα με τον περιορισμό $j > k > i$, έτσι σύμφωνα με την αρχική υπόθεση $c^1 \ge c^2 \ge \cdots \ge c^m$ θα έχουμε ότι $c^j < c^k < c^i$. Επίσης έχουμε τον περιορισμό ότι ο καθένας μπορεί να επιλέξει συνδέσεις για τις οποίες η πιθανότητα για επιλογή τους από τον αντίστοιχο παίκτη είναι θετική(δηλαδή οι αντίστοιχες συνδέσεις ανήκουν στο support του):

Πίνακας Α-4

Παίκτης 1(σύνδεση j)	**Παίκτης 3(σύνδεση k)**	**Παίκτης 2(σύνδεση i)**
γ	ε^k	ζ^i, γ^i
	δ^k	$\zeta^i, \delta^i, \gamma^i$
	γ^k	γ^i
ζ^j	γ^k	γ^i
ε^j	ε^k	ζ^i, γ^i
	γ^k	γ^i
γ^j	γ^k	γ^i

Παρατηρούμε ότι προκύπτουν 11 περιπτώσεις οι οποίες αποτελούν και τον τέταρτο όρο.

Ανάλυση του όρου $\sum_{1\le i<k\le m} p_3^k\, p_2^i\, p_1^j \left(\frac{2}{c^i} - \frac{1}{c^k}\right)$ στην εφαρμογή του λήμματος 4.1 στην περίπτωση της αυθαίρετης Ισορροπίας Nash :

Ο παίκτης 1 επιλέγει την σύνδεση j,ο παίκτης 2 την σύνδεση i και ο παίκτης 3 την σύνδεση k. Οι παίκτες επιλέγουν συνδέσεις σύμφωνα με τον περιορισμό $i < k$(όπου $i = j$), έτσι σύμφωνα με την αρχική υπόθεση $c^1 \ge c^2 \ge \cdots \ge c^m$ θα έχουμε ότι $c^k < c^i$. Επίσης έχουμε τον περιορισμό ότι ο καθένας μπορεί να επιλέξει συνδέσεις για τις οποίες η πιθανότητα για επιλογή τους από τον αντίστοιχο παίκτη είναι θετική(δηλαδή οι αντίστοιχες συνδέσεις ανήκουν στο support του):

Πίνακας Α-5

Παίκτης 3(σύνδεση k)	**Παίκτης 1-2(σύνδεση $i = j$)**
α	ζ^i, γ^i
ε^k	ζ^i, γ^i
δ^k	ζ^i, γ^i
γ^k	γ^i

Παρατηρούμε ότι προκύπτουν 7 περιπτώσεις οι οποίες αποτελούν και τον πέμπτο όρο.

Ανάλυση του όρου $\sum_{1 \le k < i \le m} p_3^k \, p_2^i \, p_1^j \left(\frac{2}{c^i} - \frac{1}{c^k}\right)$ στην εφαρμογή του λήμματος 4.1 στην περίπτωση της αυθαίρετης Ισορροπίας Nash :

Ο παίκτης 1 επιλέγει την σύνδεση j,ο παίκτης 2 την σύνδεση i και ο παίκτης 3 την σύνδεση k. Οι παίκτες επιλέγουν συνδέσεις σύμφωνα με τον περιορισμό $k < i$(όπου $i = j$), έτσι σύμφωνα με την αρχική υπόθεση $c^1 \ge c^2 \ge \cdots \ge c^m$ θα έχουμε ότι $c^k > c^i$. Επίσης έχουμε τον περιορισμό ότι ο καθένας μπορεί να επιλέξει συνδέσεις για τις οποίες η πιθανότητα για επιλογή τους από τον αντίστοιχο παίκτη είναι θετική(δηλαδή οι αντίστοιχες συνδέσεις ανήκουν στο support του):

Πίνακας Α-6

Παίκτης 1-2(σύνδεση $i = j$)	**Παίκτης 3(σύνδεση k)**
ζ^i	γ^k
γ^i	γ^k

Παρατηρούμε ότι προκύπτουν 2 περιπτώσεις οι οποίες αποτελούν και τον έκτο όρο.

Παράρτημα Β

Ανάλυση του όρου $\sum_{1\le k<j<i\le m} p_3^k\, p_2^i\, p_1^j \left(\frac{1}{c^i}-\frac{1}{c^k}\right)$ στην εφαρμογή του λήμματος 4.1 στην περίπτωση της Πλήρως Μικτής Ισορροπίας Nash :

Ο παίκτης 1 επιλέγει την σύνδεση j,ο παίκτης 2 την σύνδεση i και ο παίκτης 3 την σύνδεση k. Οι παίκτες επιλέγουν συνδέσεις σύμφωνα με τον περιορισμό $i > j > k$, έτσι σύμφωνα με την αρχική υπόθεση $c^1 \ge c^2 \ge \cdots \ge c^m$ θα έχουμε ότι $c^i < c^j < c^k$. Τώρα όλες οι συνδέσεις έχουν θετική πιθανότητα να επιλεγούν από όλους τους χρήστες.

Πίνακας Β-1

Παίκτης 2(σύνδεση i)	**Παίκτης 1(σύνδεση j)**	**Παίκτης 3(σύνδεση k)**
β	γ	$\delta^k, \varepsilon^k, \zeta^k, \gamma^k$
	δ^j	$\delta^k, \varepsilon^k, \zeta^k, \gamma^k$
	ε^j	$\varepsilon^k, \zeta^k, \gamma^k$
	ζ^j	ζ^k, γ^k
	γ^j	γ^k
ζ^i	ζ^j	ζ^k, γ^k
	γ^j	γ^k
δ^i	δ^j	$\delta^k, \varepsilon^k, \zeta^k, \gamma^k$
	ε^j	$\varepsilon^k, \zeta^k, \gamma^k$
	ζ^j	ζ^k, γ^k
	γ^j	γ^k
ε^i	ε^j	$\varepsilon^k, \zeta^k, \gamma^k$
	ζ^j	ζ^k, γ^k
	γ^j	γ^k
α	β	$\gamma, \delta^k, \varepsilon^k, \zeta^k, \gamma^k$
	γ	$\delta^k, \varepsilon^k, \zeta^k, \gamma^k$
	ζ^j	ζ^k, γ^k
	ε^j	$\varepsilon^k, \zeta^k, \gamma^k$
	δ^j	$\delta^k, \varepsilon^k, \zeta^k, \gamma^k$
	γ^j	γ^k

γ	δ^j	$\delta^k, \varepsilon^k, \zeta^k, \gamma^k$
	ε^j	$\varepsilon^k, \zeta^k, \gamma^k$
	ζ^j	ζ^k, γ^k
	γ^j	γ^k
γ^i	γ^j	γ^k

Παρατηρούμε ότι προκύπτουν 63 περιπτώσεις οι οποίες αποτελούν και τον πρώτο όρο.

Ανάλυση του όρου $\sum_{1\le j<k<i\le m} p_3^k\, p_2^i\, p_1^j \left(\frac{1}{c^i} - \frac{1}{c^k}\right)$ στην εφαρμογή του λήμματος 4.1 στην περίπτωση της Πλήρως Μικτής Ισορροπίας Nash :

Ο παίκτης 1 επιλέγει την σύνδεση j,ο παίκτης 2 την σύνδεση i και ο παίκτης 3 την σύνδεση k. Οι παίκτες επιλέγουν συνδέσεις σύμφωνα με τον περιορισμό $i > k > j$, έτσι σύμφωνα με την αρχική υπόθεση $c^1 \ge c^2 \ge \cdots \ge c^m$ θα έχουμε ότι $c^i < c^k < c^j$. Τώρα όλες οι συνδέσεις έχουν θετική πιθανότητα να επιλεγούν από όλους τους χρήστες.

Πίνακας Β-2

Παίκτης 2(σύνδεση *i*)	**Παίκτης 3(σύνδεση *k*)**	**Παίκτης 1(σύνδεση *j*)**
β	γ	$\delta^j, \varepsilon^j, \zeta^j, \gamma^j$
	δ^k	$\delta^j, \varepsilon^j, \zeta^j, \gamma^j$
	ε^k	$\varepsilon^j, \zeta^j, \gamma^j$
	ζ^k	ζ^j, γ^j
	γ^k	γ^j
ζ^i	ζ^k	ζ^j, γ^j
	γ^k	γ^j
δ^i	δ^k	$\delta^j, \varepsilon^j, \zeta^j, \gamma^j$
	ε^k	$\varepsilon^j, \zeta^j, \gamma^j$
	ζ^k	ζ^j, γ^j
	γ^k	γ^j
ε^i	ε^k	$\varepsilon^j, \zeta^j, \gamma^j$
	ζ^k	ζ^j, γ^j
	γ^k	γ^j

α	β	$\gamma, \delta^j, \varepsilon^j, \zeta^j, \gamma^j$
	γ	$\delta^j, \varepsilon^j, \zeta^j, \gamma^j$
	ζ^k	ζ^j, γ^j
	ε^k	$\varepsilon^j, \zeta^j, \gamma^j$
	δ^k	$\delta^j, \varepsilon^j, \zeta^j, \gamma^j$
	γ^k	γ^j
γ	δ^k	$\delta^j, \varepsilon^j, \zeta^j, \gamma^j$
	ε^k	$\varepsilon^j, \zeta^j, \gamma^j$
	ζ^k	ζ^j, γ^j
	γ^k	γ^j
γ^i	γ^k	γ^j

Παρατηρούμε ότι προκύπτουν 63 περιπτώσεις οι οποίες αποτελούν και το δεύτερο όρο.

Ανάλυση του όρου $\sum_{1\le k<i<j\le m} p_3^k\, p_2^i\, p_1^j \left(\frac{1}{c^j} - \frac{1}{c^k}\right)$ στην εφαρμογή του λήμματος 4.1 στην περίπτωση της Πλήρως Μικτής Ισορροπίας Nash :

Ο παίκτης 1 επιλέγει την σύνδεση j,ο παίκτης 2 την σύνδεση i και ο παίκτης 3 την σύνδεση k. Οι παίκτες επιλέγουν συνδέσεις σύμφωνα με τον περιορισμό $j > i > k$, έτσι σύμφωνα με την αρχική υπόθεση $c^1 \ge c^2 \ge \cdots \ge c^m$ θα έχουμε ότι $c^j < c^i < c^k$. Τώρα όλες οι συνδέσεις έχουν θετική πιθανότητα να επιλεγούν από όλους τους χρήστες.

Πίνακας Β-3

Παίκτης 1(σύνδεση j)	**Παίκτης 2(σύνδεση i)**	**Παίκτης 3(σύνδεση k)**
β	γ	$\delta^k, \varepsilon^k, \zeta^k, \gamma^k$
	δ^i	$\delta^k, \varepsilon^k, \zeta^k, \gamma^k$
	ε^i	$\varepsilon^k, \zeta^k, \gamma^k$
	ζ^i	ζ^k, γ^k
	γ^i	γ^k
ζ^j	ζ^i	ζ^k, γ^k
	γ^i	γ^k
δ^j	δ^i	$\delta^k, \varepsilon^k, \zeta^k, \gamma^k$

	ε^i	$\varepsilon^k, \zeta^k, \gamma^k$
	ζ^i	ζ^k, γ^k
	γ^i	γ^k
ε^j	ε^i	$\varepsilon^k, \zeta^k, \gamma^k$
	ζ^i	ζ^k, γ^k
	γ^i	γ^k
α	β	$\gamma, \delta^k, \varepsilon^k, \zeta^k, \gamma^k$
	γ	$\delta^k, \varepsilon^k, \zeta^k, \gamma^k$
	ζ^i	ζ^k, γ^k
	ε^i	$\varepsilon^k, \zeta^k, \gamma^k$
	δ^i	$\delta^k, \varepsilon^k, \zeta^k, \gamma^k$
	γ^i	γ^k
γ	δ^i	$\delta^k, \varepsilon^k, \zeta^k, \gamma^k$
	ε^i	$\varepsilon^k, \zeta^k, \gamma^k$
	ζ^i	ζ^k, γ^k
	γ^i	γ^k
γ^j	γ^i	γ^k

Παρατηρούμε ότι προκύπτουν 63 περιπτώσεις οι οποίες αποτελούν και τον τρίτο όρο.

Ανάλυση του όρου $\sum_{1\le i<k<j\le m} p_3^k\, p_2^i\, p_1^j \left(\frac{1}{c^j} - \frac{1}{c^k}\right)$ στην εφαρμογή του λήμματος 4.1 στην περίπτωση της Πλήρως Μικτής Ισορροπίας Nash :

Ο παίκτης 1 επιλέγει την σύνδεση j,ο παίκτης 2 την σύνδεση i και ο παίκτης 3 την σύνδεση k. Οι παίκτες επιλέγουν συνδέσεις σύμφωνα με τον περιορισμό $j > k > i$, έτσι σύμφωνα με την αρχική υπόθεση $c^1 \geq c^2 \geq \cdots \geq c^m$ θα έχουμε ότι $c^j < c^k < c^i$. Τώρα όλες οι συνδέσεις έχουν θετική πιθανότητα να επιλεγούν από όλους τους χρήστες.

Πίνακας B-4

Παίκτης 1(σύνδεση j)	Παίκτης 3(σύνδεση k)	Παίκτης 2(σύνδεση i)
β	γ	$\delta^i, \varepsilon^i, \zeta^i, \gamma^i$
	δ^k	$\delta^i, \varepsilon^i, \zeta^i, \gamma^i$
	ε^k	$\varepsilon^i, \zeta^i, \gamma^i$

	ζ^k	ζ^i, γ^i
	γ^k	γ^i
ζ^j	ζ^k	ζ^i, γ^i
	γ^k	γ^i
δ^j	δ^k	$\delta^i, \varepsilon^i, \zeta^i, \gamma^i$
	ε^k	$\varepsilon^i, \zeta^i, \gamma^i$
	ζ^k	ζ^i, γ^i
	γ^k	γ^i
ε^j	ε^k	$\varepsilon^i, \zeta^i, \gamma^i$
	ζ^k	ζ^i, γ^i
	γ^k	γ^i
α	β	$\gamma, \delta^i, \varepsilon^i, \zeta^i, \gamma^i$
	γ	$\delta^i, \varepsilon^i, \zeta^i, \gamma^i$
	ζ^k	ζ^i, γ^i
	ε^k	$\varepsilon^i, \zeta^i, \gamma^i$
	δ^k	$\delta^i, \varepsilon^i, \zeta^i, \gamma^i$
	γ^k	γ^i
γ	δ^k	$\delta^i, \varepsilon^i, \zeta^i, \gamma^i$
	ε^k	$\varepsilon^i, \zeta^i, \gamma^i$
	ζ^k	ζ^i, γ^i
	γ^k	γ^i
γ^j	γ^k	γ^i

Παρατηρούμε ότι προκύπτουν 63 περιπτώσεις οι οποίες αποτελούν και τον τέταρτο όρο.

Ανάλυση του όρου $\sum_{1\leq i<k\leq m} p_3^k\, p_2^i\, p_1^j \left(\frac{2}{c^i} - \frac{1}{c^k}\right)$ στην εφαρμογή του λήμματος 4.1 στην περίπτωση της αυθαίρετης Ισορροπίας Nash :

Ο παίκτης 1 επιλέγει την σύνδεση j,ο παίκτης 2 την σύνδεση i και ο παίκτης 3 την σύνδεση k. Οι παίκτες επιλέγουν συνδέσεις σύμφωνα με τον περιορισμό $i < k$(όπου $i = j$), έτσι σύμφωνα με την αρχική υπόθεση $c^1 \geq c^2 \geq \cdots \geq c^m$ θα έχουμε ότι $c^k < c^i$. Επίσης έχουμε τον περιορισμό ότι ο καθένας μπορεί να επιλέξει συνδέσεις για τις οποίες η πιθανότητα για επιλογή τους από τον αντίστοιχο παίκτη είναι θετική(δηλαδή οι αντίστοιχες συνδέσεις ανήκουν στο support του):

Πίνακας Β-5

Παίκτης 3(σύνδεση k)	Παίκτης 1-2(σύνδεση $i=j$)
α	β
	γ
	ζ^i
	ε^i
	δ^i
	γ^i
β	γ
	ζ^i
	ε^i
	δ^i
	γ^i
γ	ζ^i
	ε^i
	δ^i
	γ^i
ζ^k	ζ^i
	γ^i
ε^k	ζ^i
	ε^i
	γ^i
δ^k	ζ^i
	ε^i
	δ^i
	γ^i
γ^k	γ^i

Παρατηρούμε ότι προκύπτουν 25 περιπτώσεις οι οποίες αποτελούν και τον πέμπτο όρο.

Ανάλυση του όρου $\sum_{1 \leq k < i \leq m} p_3^k \, p_2^i \, p_1^j \left(\frac{2}{c^i} - \frac{1}{c^k}\right)$ στην εφαρμογή του λήμματος 4.1 στην περίπτωση της αυθαίρετης Ισορροπίας Nash :

Ο παίκτης 1 επιλέγει την σύνδεση j,ο παίκτης 2 την σύνδεση i και ο παίκτης 3 την σύνδεση k. Οι παίκτες επιλέγουν συνδέσεις σύμφωνα με τον περιορισμό $k < i$(όπου $i = j$), έτσι σύμφωνα με την αρχική υπόθεση $c^1 \geq c^2 \geq \cdots \geq c^m$ θα έχουμε ότι $c^i < c^k$. Επίσης έχουμε τον περιορισμό ότι ο καθένας μπορεί να επιλέξει συνδέσεις για τις οποίες η πιθανότητα για επιλογή τους από τον αντίστοιχο παίκτη είναι θετική(δηλαδή οι αντίστοιχες συνδέσεις ανήκουν στο support του):

Πίνακας Β-6

Παίκτης 1-2(σύνδεση $\boldsymbol{i = j}$)	**Παίκτης 3(σύνδεση $\boldsymbol{k}$)**
α	β
	γ
	ζ^k
	ε^k
	δ^k
	γ^k
β	γ
	ζ^k
	ε^k
	δ^k
	γ^k
γ	ζ^k
	ε^k
	δ^k
	γ^k
ζ^i	ζ^k
	γ^k
ε^i	ζ^k
	ε^k
	γ^k

δ^i	ζ^k
	ε^k
	δ^k
	γ^k
γ^i	γ^k

Παρατηρούμε ότι προκύπτουν 25 περιπτώσεις οι οποίες αποτελούν και τον έκτο όρο.